Formal Methods Laboratory
Romanian Academy

FML Technical Report

TR no. FML-15-01 **Printed: April 2015**

Consistence of Choice Principles in Finitely Supported Mathematics

A. Alexandru and G.Ciobanu

Approved for public release; further dissemination unlimited.

Prepared by
Formal Methods Laboratory.

Series of
Technical Reports (Institute of Computer Science Iaşi)

ISSN 1842 - 1490

FML is the Formal Methods Laboratory from the Institute of Computer Science of the Romanian Academy and is located in Iaşi. It is a research institution aimed at developing new formalisms for challenging open problems in computer science, systems biology and other emerging research fields. FML is headed by Dr. Gabriel Ciobanu.

IIT is the Institute of Computer Science of the Romanian Academy and is concerned with basic research in computer science.

Romanian Academy of Science is the supreme forum of science from Romania.
Calea Victoriei 125, Sector 1, 010071, Bucharest, ROMANIA.
Telephone: +40 21 2128640
Telefax: +40 21 2116608
Web: http://www.academiaromana.ro/

Printed in Romania and reproduced directly from the authors original copy.

Copyright: ©2005 Formal Methods Laboratory Iaşi (FML)
Blvd. Carol I, nr.8, 700505, Iaşi, ROMANIA (RO)
Institute of Computer Science, Romanian Academy, branch Iaşi.
Telephone/Telefax: +40 232 241708
Web: http://iit.iit.tuiasi.ro/fml
Contact: gabriel@iit.tuiasi.ro

Notice: Permission is hereby granted for the redistribution of this material so long as this item is redistributed in full and with appropriate credit given to the author(s). All other rights reserved.

Series of Technical Reports (Institute of Computer Science Iaşi)

ISSN 1842 - 1490

Consistence of Choice Principles in Finitely Supported Mathematics

by

Andrei Alexandru
Institute of Computer Science,
Romanian Academy, Iaşi
aalexandru@iit.tuiasi.ro

Gabriel Ciobanu
Institute of Computer Science,
Romanian Academy, Iaşi
gabriel@iit.tuiasi.ro

ABSTRACT

Finitely Supported Mathematics represents a part of mathematics for experimental sciences which has a continuous evolution in the last century. It is developed according to the Fraenkel-Mostowski axioms of set theory. The axiom of choice is inconsistent in the Finitely Supported Mathematics. We prove that several weaker forms of the axiom of choice are also inconsistent in the Finitely Supported Mathematics.

Contents

1 Introduction

Since the experimental sciences are mainly interested in quantitative aspects, and since there exists no evidence for the presence of infinite structures, it becomes useful to develop a mathematics which deals with a more relaxed notion of (in)finiteness. What we call Finitely Supported Mathematics is a mathematics which is consistent with the axioms of the Fraenkel-Mostowski (FM) set theory. The FM axioms represents an "axiomatization" of the FM permutation model of the Zermelo-Fraenkel set theory with atoms; in this way, these axioms transform this model into an independent set theory. The axioms of the FM set theory are precisely the Zermelo-Fraenkel with atoms (ZFA) axioms over an infinite set of atoms [21], together with the special property of finite support which claims that for each element x in an arbitrary set we can find a finite set supporting x. Therefore in the FM universe only finitely supported objects are allowed. The original purpose of the axiomatic FM set theory was to provide a mathematical model for variables in a certain syntax. Since they have no internal structure, atoms can be used to represent names. The finite support axiom is motivated by the fact that syntax can only involve finitely many names. The FM set theory provides a balance between rigorous formalism and informal reasoning. This is discussed in [32], where principles of structural recursion and induction are explained in the FM framework. We can use this theory in order to manage infinite structures in a finite manner, that is, in the FM framework we try to model the infinite using a more relaxed notion of finite, i.e, the notion of finite support.

The construction of the universe of all FM-sets [21] is inspired by the construction of the universe of all admissible sets over an arbitrary collection of atoms [7]. The FM-sets represent a generalization of hereditary finite sets (which are particular admissible sets); actually, any FM-set is an hereditary finitely supported set.

In the literature there exist various approaches regarding the FM framework that have a continuous evolution during the last century. We try to clarify the differences between these approaches. Such a classification have also been pointed in [1].

- **The FM permutation model of the ZFA set theory.** This model was introduced by Fraenkel [18] and extended by Lindenbaum and Mostowski [30]. Its original aim was to establish the independence of the axiom of choice from the other axioms of the ZFA set theory. There also exist some other permutation models of ZFA presented in [28] which are defined by using countable infinite sets of atoms.

- **The FM axiomatic set theory.** This set theory was presented in [21]. It is inspired by the FM permutation model of the ZFA set theory. However, the FM set theory, the ZFA set theory and the Zermelo-Fraenkel (ZF) set theory are independent axiomatic set theories. All of these theories are described by axioms, and all of them have models. For example, the Cumulative Hierarchy Fraenkel-Mostowski universe $FM(A)$ presented in [21] is a model of the FM set theory, while some models of the ZF set theory can be found in [27], and the permutation models of the ZFA set theory can be found in [28]. The sets defined using the FM axioms are called FM-sets. A ZFA set is an FM-set if and only if all its elements have hereditary finite supports. Note that the infinite set of atoms in the FM set theory does not necessary be countable. The Fraenkel-Mostowski set theory is consistent whether the infinite set of atoms is countable or not. In [21] it is used a countable set of atoms in order to define a model of the Fraenkel-Mostowski set theory for new names in computer science, while in [8] there are described FM-sets over a set of atoms which do not represent a homogeneous structure. Also, in [13] the authors use non-countable sets of atoms (like the set of real numbers) in order to study the minimization of deterministic timed automata.

- **Nominal sets.** These sets can be defined both in the ZF framework [33] and in the FM framework [21]. In ZF, a fixed infinite set A is considered as a set of names. A nominal set is defined as a usual ZF set endowed with a particular group action of the group of permutations over A that satisfies a certain finiteness property. Such a finiteness property allows us to say that nominal sets are well defined according to the axioms of the FM set theory whenever the set of names is the set of atoms in the FM set theory. There exists also an alternative definition for nominal sets in the FM framework. They can be defined as sets constructed according to the FM axioms with the additional property of being empty supported (invariant under all permutations). These two ways of defining nominal sets finally lead to similar properties. According to the previous remark we use the terminology "invariant" for "nominal" in order to establish a connection between approaches in the FM framework and in the ZF framework. Moreover, we can say that any set defined according to the FM axioms (any FM-set) can be seen as a subset of the nominal (invariant) set $FM(A)$. However, an FM-set is itself a nominal set only if it has an empty support. The theory of nominal sets makes sense even if the set of atoms is infinite but not countable. Informally, since the ZFA set theory collapses into the ZF set theory when the set of atoms is empty, we can say that the nominal sets represent a natural extension of the usual sets. In computer science, nominal sets offer an elegant formalism for describing λ-terms modulo α-conversion [21]. They can also be used in algebra [5, 2], in proof theory [38], in domain theory [37], in topology [31], semantics of process algebras [4, 19] and programming [36]. A survey on the applications of nominal sets in computer science emphasizing our contributions can be found in [3].

- **Generalized nominal sets.** The theory of nominal sets over a fixed set A of atoms is generalized in [11] to a new theory of nominal sets over arbitrary (unfixed) sets of data values. This provides the generalized nominal sets. The notion of 'S_A-set' (Definition 3.2) is replaced by the notion of 'set endowed with an action of a subgroup of the symmetric group of $\mathbb{D}$' for an arbitrary set of data values $\mathbb{D}$, and the notion of 'finite set' is replaced by the notion of 'set with a finite number of orbits according to the previous group action (orbit-finite set)'. This approach is useful for studying automata on data words [11], languages over infinite alphabets [9], or Turing machines that operate over infinite alphabets [12]. Computations in these generalized nominal sets are presented in [10, 14].

As their names say, the nominal sets are used to manage notions like renaming, binding or fresh name. However, this theory could be studied deeper from an algebraically viewpoint, and it could be used in order to characterize some infinite structures in terms of finitely supported objects.

Finitely Supported Mathematics (FSM) is introduced in [1] in order to prove that many finiteness ZF properties still remain valid if we replace the term 'finite' with 'infinite, but with finite support'. Such results have already been presented in [5] where we proved that a class of multisets over infinite alphabets (interpreted in the nominal framework) has similar properties to the classical multisets over finite alphabets. FSM is the mathematics developed in the world of finitely supported objects where the set of atoms has to be infinite (countable or not countable). Informally, FSM extends the framework of the ZF set theory without choice principles; ZF set theory is actually the Empty Supported Mathematics. In FSM, we use either 'invariant sets' or 'finitely supported sets' instead of 'sets'. As an intuitive rule, we are not allowed to use in the proofs of the results of FSM any construction that does not preserve the property of finite support. That means we cannot obtain a property in FSM only by using a ZF result without an appropriate proof using only the finite support condition. Since the invariant sets can also be defined in the ZFA framework similarly as in the ZF framework, the definition of the finitely supported mathematics also makes sense over the ZFA axioms.

To summarize, FSM represents the ZF theory rephrased in terms of finitely supported objects; this means that FSM presents the theory of invariant sets, including invariant algebraic structures. FSM is not at all the theory of nominal sets from [33] presented in a different manner; actually the theory of nominal sets [33] could be considered as a tool for defining FSM. The main aim of FSM is to characterize the infinite algebraic structures by using their finite supports.

The FM set theory was constructed initially in 1930s, in order to prove independence of the axiom of choice and other axioms in the classical ZF set theory. Later, in 2000s, the FM set theory found a lot of applications in experimental sciences. However, the consistence of the various weaker forms of choice (which were proved to be independent from the axioms of ZF, in the last century) with the new defined axioms of the FM set theory remained an open problem. The FM set theory have a continuous evolution in the last century starting with the permutation models of ZFA and finishing with FSM. Our goal is to study the consistence of various choice principles internally in FSM. It is well known, from 1930s, that the full axiom of choice is inconsistent in FSM. However, there does not exist yet results regarding the consistency of various weaker forms of choice in FSM. The aim of this paper is to study whether some weaker forms of the axiom of choice can be consistent in FSM. Also, several order and choice properties of nominal (invariant) sets are obtained by comparing various choice principles in FSM. Details regarding this aspect can be found in Section 5. This paper was announced in the conference paper [1] as a future work. Actually, this paper extends [1] by presenting some examples of ZF results that cannot be translated into FSM.

2 Axiom of Choice and Choice Principles in the Zermelo-Fraenkel Framework

The Axiom of Choice (**AC**) is probably the most discussed axiom in mathematics after Euclid's Axiom of Parallels. The first formulation of **AC** is due to Zermelo [39]. It claims that given any family of non-empty sets $\mathcal{F}$, it is possible to select a single element from each member of $\mathcal{F}$. This statement is equivalent to the assertion that for any family of non-empty sets $\mathcal{F}$, there exists at least one choice function on $\mathcal{F}$, where a choice function on $\mathcal{F}$ is a function f with domain $\mathcal{F}$ such that, for each non-empty set X in $\mathcal{F}$, $f(X)$ is an element of X. Zermelo's purpose in introducing **AC** was to establish a central principle of Cantor's set theory, namely, that every set admits a well-ordering and so can also be assigned a cardinal number. There are more than 200 mathematical results which are proved to be equivalent to the axiom of choice (see [28] and [35]) in the ZF (with the axiom of foundation) set theory. The most common are:

- The Cartesian product of a family of non-empty sets is non-empty;
- Every surjective function has a right inverse;
- For any family $(X_i)_{i\in I}$ of non-empty sets there exists a family $(F_i)_{i\in I}$ of non-empty, finite sets F_i with $F_i \subseteq X_i$ for each $i \in I$ (Axiom of multiple choice);
- Any non-empty inductive poset P (i.e. any poset P for which every chain in P has an upper bound in P) has a maximal element (Zorn's lemma **ZL**);
- Each partially ordered set contains a maximal totally ordered subset (Hausdorff's maximal principle);
- In the product topology, the closure of a product of subsets is equal to the product of the closures;
- Every partially ordered set has a maximal subset such that any two elements in the subset are incomparable (antichain principle);
- Every set can be well-ordered (Zermelo's well-ordering theorem);
- In any pair of cardinal numbers, one is less than the other, or they are equal (Trichotomy principle);
- The product of any family of compact topological spaces is compact (Tychonov's theorem);
- Every vector space has a basis (Hamel's theorem);
- For every infinite set X, there exists a bijective map between the sets X and $X \times X$ (Tarski's theorem);
- Every non-trivial ring $(R, +, \cdot, 1)$ contains a maximal ideal;
- Every lattice with a largest element has a maximal ideal;
- Every divisible module over a principal ideal domain is injective;
- Every infinite consistent set Σ of first-order sentences has a model of cardinality no greater than that of Σ (the model existence theorem for first-order logic).

Some weaker forms of the axiom of choice, also called choice principles, are collected in [26]:

- **Axiom of dependent choice (DC)**: let R be a non-empty relation on a set X with the property that for each $x \in X$ there exists $y \in X$ with xRy. Then there exists a function $f : \omega \to X$ such that $f(n)Rf(n+1)$, $\forall n \in \omega$;
- **Axiom of countable choice (CC)**: given any countable family (sequence) of non-empty sets $\mathcal{F}$, it is possible to select a single element from each member of $\mathcal{F}$;
- **Axiom of partial countable choice (PCC)**: given any countable family (sequence) of non-empty sets $\mathcal{F} = (X_n)_n$, there exists an infinite subset M of $\mathbb{N}$ such that it is possible to select a single element from each member of the family $(X_m)_{m \in M}$;
- **Axiom choice over finite sets (AC(fin))**: given any family of finite non-empty sets $\mathcal{F}$, it is possible to select a single element from each member of $\mathcal{F}$;
- **Axiom countable choice over finite sets (CC(fin))**: given any countable family (sequence) of finite non-empty sets $\mathcal{F}$, it is possible to select a single element from each member of $\mathcal{F}$;
- **Axiom of partial countable choice over finite sets (PCC(fin))**: given any countable family (sequence) of non-empty finite sets $\mathcal{F} = (X_n)_n$, there exists an infinite subset M of $\mathbb{N}$ such that it is possible to select a single element from each member of the family $(X_m)_{m \in M}$
- **Boolean prime ideal theorem (PIT)**: every boolean algebra with $0 \neq 1$ has a maximal ideal (and hence a prime ideal);
- **Boolean ultrafilter theorem (UFT)**: in a Boolean algebra, every filter can be enlarged to a maximal one;
- **Compactness of products of Hausdorff spaces (CPHS)**: products of compact Hausdorff spaces are compact;
- **Kinna-Wagner Selection Principle (KW)**: given any family $\mathcal{F}$ of sets of cardinality at least 2, there exists a function f on $\mathcal{F}$ such that $f(X)$ is a non-empty proper subset of X for each $X \in \mathcal{F}$;
- **Refinement of the Kinna-Wagner Selection Principle (RKW)**: given any set X and the family $\mathcal{F}$ of all subsets of X of cardinality at least 2, there exists a function f on $\mathcal{F}$ such that $f(Y)$ is a non-empty proper subset of Y for each $Y \in \mathcal{F}$;
- **Ordering principle (OP)**: every set can be totally ordered;
- **Order extension principle (OEP)**: every partial order relation on a set can be enlarged to a total order relation;
- **Axiom of Dedekind infiniteness (Fin)** every infinite set X allows an injection $i : \mathbb{N} \to X$.

According to [26] we have the following theorem:

THEOREM 2.1 *The following implications are valid in the ZF set theory:*

1. ***AC*** $\Rightarrow$ ***DC*** $\Rightarrow$ ***CC*** $\Leftrightarrow$ ***PCC*** $\Rightarrow$ ***CC(fin)****;*

2. ***PIT*** $\Leftrightarrow$ ***UFT*** $\Leftrightarrow$ ***CPHS****;*

3. ***AC*** $\Rightarrow$ ***UFT*** $\Rightarrow$ ***OEP*** $\Rightarrow$ ***OP*** $\Rightarrow$ ***AC(fin)*** $\Rightarrow$ ***CC(fin)****;*

4. ***CC*** $\Rightarrow$ ***Fin*** $\Rightarrow$ ***PCC(fin)*** $\Leftrightarrow$ ***CC(fin)****;*

5. ***KW*** $\Leftrightarrow$ ***RKW*** $\Rightarrow$ ***OP*** [1].

An important connection between **AC** and mathematical logic was established by Goodman and Myhill [23]:

THEOREM 2.2 ***AC*** *implies the Law of Excluded Middle which states that for any proposition, either that proposition is true, or its negation is true.*

There exists a large number of results which cannot be proved without employing **AC** or the previous weaker versions of **AC**. We remind some of them. More details are and complete proofs can be found in [25], [26], [28], [34] and [35]. We specify for each result which is the *minimal* form of choice requested for its proof.

- The following definitions of finiteness are equivalent in the ZF set theory with **AC**:

    1. X is finite if it corresponds one-to-one and onto to a finite ordinal (usual finiteness);
    2. X is finite if it has no injections into any of its proper subsets (Dedekind-Pierce finiteness);
    3. $\emptyset$ is finite, $\{x\}$ is finite for each x, if X and Y are finite then so is $X \cup Y$, and no other sets are finite (Kuratowski finiteness);
    4. X is finite if every total ordering on it is a well-ordering (Tarski finiteness);
    5. X is finite if for all increasing chains $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$ such that $X \subseteq \cup X_n$ there exists $n \in \mathbb{N}$ such that $X \subseteq X_n$;
    6. X is finite if for all directed collections $(X_i)_{i \in I}$ such that $X \subseteq \cup X_i$ there exists $i \in I$ such that $X \subseteq X_i$.

- Any union of countably many countable sets is itself countable – the proof requires **CC**;

- There exists a subset of the real numbers that is not Lebesgue measurable (Vitali's theorem) –the proof requires **PIT**;

- In a metric space the notions of compactness and sequential compactness are equivalent (Bolzano-Weierstrass theorem) – the proof requires **CC**;

[1]The first equivalence follows obviously, and the second implication follows from the proof of Theorem 4.40 in [26].

- Countable sums of Lindelof spaces are Lindelof – the proof requires **CC**;
- Every submodule of a free module over a principal ideal domain is also free (Kaplansky's theorem) – the proof requires **ZL**;
- Every free left module is projective – the proof requires **AC**;
- Every field has an algebraic closure – the proof is a consequence of **PIT**;
- Every field extension has a transcendence basis – the proof requires **ZL**;
- Any automorphism of a subfield of an algebraically closed field K can be extended to the whole of K – the proof requires **ZL**;
- All the bases of a vector space have the same cardinality –the proof requires **ZL**;
- Every linear functional f defined on a subspace U of a real vector space V which is dominated by the sublinear function $p : V \to \mathbb{R}$ has a linear extension $F : V \to \mathbb{R}$ with the property that F is dominated by p on V (Hahn-Banach theorem) – the proof requires a weaker form of **PIT** which states that on every Boolean algebra there exists an additive, normed $[0, 1]$-valued measure;
- Every Hilbert space has an orthonormal basis – the proof requires **ZL**;
- Let $(X, \leq)$ be a poset, and $S : X \to \mathbb{R} \cup \{\infty\}$ a function. We assume that the following conditions are satisfied:

  1. Each increasing sequence (x_n) in X with the property that $(S(x_n))$ is strictly increasing, is bounded;
  2. The function S is increasing.

  Then for each $x_0 \in X$ there exists $\overline{x} \in X$ such that $x_0 \leq \overline{x}$ and $S(x) = S(\overline{x})$ for each x with the property that $\overline{x} \leq x$. (Brezis - Browder theorem) – the proof requires **DC**;
- Every complete metric space is a Baire space i.e. a topological space with the property that the union of any countable collection of closed sets with empty interior has empty interior (Baire's category theorem) – the proof requires **DC**;
- Every consistent set of first-order sentences can be extended to a maximal consistent set (Gödel's completeness theorem) – the proof requires **PIT**;
- If every finite subset of a of a set of first-order sentences has a model, then the set has a model (Compactness theorem for first-order logic) – the proof requires **PIT**.

Many areas in mathematics, especially in functional analysis, real analysis, topology, the theory of differential equations, the theory of algebraic structures, measure theory, first-order logic, and category theory heavily depend on choice principles. The axiom of choice has a lot of elegant consequences, but that is an argument for its mathematical interest, not for its correctness. Since its first postulation the axiom of choice became the subject of many controversies. The first controversy is about the meaning of the word "exists" since this term is very vague. One group of mathematicians (called intuitionists) believes that a set exists only if each of its elements can be designated specifically or at least if there is a law by which each of its elements can be constructed. Another controversy is represented by a geometrical consequence of **AC** known as Banach and Tarski's paradoxical decomposition of the sphere. In [6] they showed that any solid sphere can be split into finitely many subsets which can themselves be reassembled to form two solid spheres, each of the same size as the original; and any solid sphere can be split into finitely many subsets in such a way as to enable them to be reassembled to form a solid sphere of arbitrary size. Questions about the **AC**'s independence of the systems of set-theoretic axioms appeared naturally. In 1922 Fraenkel introduces the permutation method to establish the independence of **AC** from a system of set theory with atoms [18]. That is, Fraenkel constructed a model in which the axioms of set theory excluding the axiom of choice are satisfied but this model contains a set which does not satisfy the axiom of choice. Fraenkel's model was refined and extended by Lindenbaum and Mostowski [30], and, later, by Gabbay and Pitts [21] to what we call the *Fraenkel-Mostowski set theory*. More precisely, the Fraenkel-Mostowski set theory was initially described as a model of the Zermelo-Fraenkel with atoms set theory. However, in [21] the Fraenkel-Mostowski set theory has been developed as an independent axiomatic set theory. In [22] Gödel proved that the axiom of choice is consistent with the other axioms of set theory (von Neumann-Bernays-Gödel set theory). He proved that given a model for set theory in which there are no atoms and the axiom of foundation is true, there exists a model in which, in addition, the axiom of choice is true. Moreover, if Gödel's model is modified so that either atoms exist or the axiom of foundation is false, the validity of the axiom of choice is not disturbed. Therefore, the axiom of choice is consistent with the other axioms of set theory regardless of whether atoms exist or not, and whether the axiom of foundation is valid or not. Fraenkel showed that the collections of sets of atoms need not necessarily have choice functions [18]. However, at that time he was unable to establish the same fact for the usual sets of mathematics, for example the set of real numbers. This problem remained unsolved until 1963 when Cohen proved the independence of **AC** (and of the axiom of countable choice) from the standard axioms of set theory [16, 17]. Cohen's independence proof (known as the method of *forcing*) also made use of permutations in essentially the form in which Fraenkel had originally employed them.

3 Invariant Sets

Let A be a fixed infinite (countable or non-countable) ZF-set. The following results make also sense if A is considered to be the set of atoms in the ZFA framework (characterized by the axiom "$y \in x \Rightarrow x \notin A$") and if 'ZF' is replaced by 'ZFA' in their statement. Thus, we mention that the theory of invariant sets makes sense both in ZF and in ZFA. The results in this section are similar to those in [33] with the mention that we do not assume the set of atoms to be countable. In Section 2 we proved that the equivalence of various definitions of finiteness is a consequence of **AC**. According to Theorem 4.3, **AC** is inconsistent with the axioms of the FM set theory. To avoid any doubt, where the word "finite" appears in this paper without reference to any of the definitions of finiteness from Section 2, it means "it corresponds one-to-one and onto to a finite ordinal".

DEFINITION 3.1 *i) A* transposition *is a function* $(a\,b) : A \to A$ *defined by* $(a\,b)(a) = b$, $(a\,b)(b) = a$ *and* $(a\,b)(n) = n$ *for* $n \neq a, b$.

ii) A permutation *of A is a one-to-one and onto function on A which interchanges only finitely many elements.*

Let S_A be the group of all permutations; in our approach S_A is not the entire set of bijections of A, but the set of those bijections of A which can be expressed by composing finitely many transpositions.

DEFINITION 3.2

- *Let X be a ZF-set. An* S_A-action *on X is a function* $\cdot : S_A \times X \to X$ *having the properties that* $Id \cdot x = x$ *and* $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ *for all* $\pi, \pi' \in S_A$ *and* $x \in X$.
- *An* S_A-set *is a pair* $(X, \cdot)$ *where X is ZF-set, and* $\cdot : S_A \times X \to X$ *is an* S_A*-action on* X. *We simply use X whenever no confusion arises.*

DEFINITION 3.3 *Let* $(X, \cdot)$ *be an* S_A*-set. We say that* $S \subset A$ supports x *whenever for each* $\pi \in Fix(S)$ *we have* $\pi \cdot x = x$, *where* $Fix(S) = \{\pi \,|\, \pi(a) = a, \forall a \in S\}$.

When for an element x in an S_A-set we can find a finite set supporting it, we also say that "x has the finite support property" or "x is finitely supported".

DEFINITION 3.4 *Let* $(X, \cdot)$ *be an* S_A*-set. We say that X is an* invariant set *if for each* $x \in X$ *there exists a finite set* $S_x \subset A$ *which supports* x. *Invariant sets are also called* nominal sets *if we work in the ZF framework [33], or* equivariant sets *if they are defined as elements in the cumulative hierarchy* $FM(A)$ *[21].*

THEOREM 3.1 ([33]) *Let X be an* S_A*-set, and for each* $x \in X$ *let us consider* $\mathcal{F}_x = \{S \subset A \,|\, S\, \text{finite}, S\, \text{supports}\, x\}$. *If* $\mathcal{F}_x$ *is non-empty (particularly if X is an invariant set), then it has a least element which also supports* x. *We call this element* the support of x, *and we denote it by* $supp(x)$.

PROPOSITION 3.2 ([33]) *Let* $(X, \cdot)$ *be an* S_A*-set and let* $\pi \in S_A$ *be an arbitrary permutation. Then for each* $x \in X$ *which is finitely supported we have that* $\pi \cdot x$ *is finitely supported and* $supp(\pi \cdot x) = \pi(supp(x))$.

EXAMPLE 3.1

1. *The set A of atoms is an* S_A*-set with the* S_A*-action* $\cdot : S_A \times A \to A$ *defined by* $\pi \cdot a := \pi(a)$ *for all* $\pi \in S_A$ *and* $a \in A$. $(A, \cdot)$ *is an invariant set because for each* $a \in A$ *we have that* $\{a\}$ *supports a. Moreover,* $supp(a) = \{a\}$ *for each* $a \in A$.

2. *The set A of atoms is an* S_A*-set with the* S_A*-action* $\cdot : S_A \times A \to A$ *defined by* $\pi \cdot a := a$ *for all* $\pi \in S_A$ *and* $a \in A$. $(A, \cdot)$ *is an invariant set because for each* $a \in A$ *we have that* $\emptyset$ *supports a. Moreover,* $supp(a) = \emptyset$ *for each* $a \in A$.

3. *The set S_A is an S_A-set with the S_A-action $\cdot : S_A \times S_A \to S_A$ defined by $\pi \cdot \sigma := \pi \circ \sigma \circ \pi^{-1}$ for all $\pi, \sigma \in S_A$. $(S_A, \cdot)$ is an invariant set because for each $\sigma \in S_A$ we have that the finite set $\{a \in A \,|\, \sigma(a) \neq a\}$ supports σ. Moreover, $supp(\sigma) = \{a \in A \,|\, \sigma(a) \neq a\}$ for each $\sigma \in S_A$.*

4. *Any ordinary ZF-set X (like $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$ for example) is an S_A-set with the S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in X$. Also X is an invariant set because for each $x \in X$ we have that $\emptyset$ supports x. Moreover, $supp(x) = \emptyset$ for each $x \in X$.*

5. *If $(X, \cdot)$ is an S_A-set then $\wp(X) = \{Y \,|\, Y \subseteq X\}$ is also an S_A-set with the S_A-action $\star : S_A \times \wp(X) \to \wp(X)$ defined by $\pi \star Y := \{\pi \cdot y \,|\, y \in Y\}$ for all $\pi \in S_A$, and all $Y \subseteq X$. Note that $\wp(X)$ does not necessarily have to be an invariant set even if X is. For each invariant set $(X, \cdot)$, we denote by $\wp_{fs}(X)$ the set formed from those subsets of X which are finitely supported according to the action $\star$. According to Proposition 3.2, $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is an invariant set, where $\star|_{\wp_{fs}(X)} : S_A \times \wp_{fs}(X) \to \wp_{fs}(X)$ is defined by $\pi \star|_{\wp_{fs}(X)} Y := \pi \star Y$ for all $\pi \in S_A$ and $Y \in \wp_{fs}(X)$.*

6. *Let $(X, \cdot)$ and $(Y, \diamond)$ be S_A-sets. As usual, we define the Cartesian product $X \times Y$ as the set of ordered pair $(x, y) = \{\{x\}, \{x, y\}\}$ for $x \in X$ and $y \in Y$. $X \times Y$ is also an S_A-set with the S_A-action $\star : S_A \times (X \times Y) \to (X \times Y)$ defined by $\pi \star (x, y) = (\pi \cdot x, \pi \diamond y)$ for all $\pi \in S_A$ and all $x \in X$, $y \in Y$. If $(X, \cdot)$ and $(Y, \diamond)$ are invariant sets, then $(X \times Y, \star)$ is also an invariant set.*

7. *Let $(X, \cdot)$ and $(Y, \diamond)$ be S_A-sets. We define the disjoint union of X and Y by $X + Y = \{(0, x) \,|\, x \in X\} \cup \{(1, y) \,|\, y \in Y\}$. $X + Y$ is an S_A-set with the S_A-action $\star : S_A \times (X+Y) \to (X+Y)$ defined by $\pi \star z = (0, \pi \cdot x)$ if $z = (0, x)$ and $\pi \star z = (1, \pi \diamond y)$ if $z = (1, y)$. If $(X, \cdot)$ and $(Y, \diamond)$ are invariant sets, then $(X+Y, \star)$ is also an invariant set: each $z \in X+Y$ is either of the form $(0, x)$ and supported by the finite set supporting x in X, or is of the form $(1, y)$ and supported by the finite set supporting y in Y.*

DEFINITION 3.5 *Let $(X, \cdot)$ be an invariant set. A subset Z of X is called* finitely supported *if and only if $Z \in \wp_{fs}(X)$.*

Recall that a function $f : X \to Y$ is a particular relation. Precisely, a function $f : X \to Y$ is a subset f of $X \times Y$ characterized by the property that for each $x \in X$ there is exactly one $y \in Y$ such that $(x, y) \in f$.

DEFINITION 3.6 *Let X and Y be invariant sets. A function $f : X \to Y$ is* finitely supported *if $f \in \wp_{fs}(X \times Y)$.*

Let $Y^X = \{f \subseteq X \times Y \mid f$ is a function from the underlying set of X to the underlying set of $Y\}$.

PROPOSITION 3.3 ([33]) *Let $(X, \cdot)$ and $(Y, \diamond)$ be invariant sets. Then Y^X is an S_A-set with the S_A-action $\star : S_A \times Y^X \to Y^X$ defined by $(\pi \star f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. A function $f : X \to Y$ is finitely supported in the sense of Definition 3.6 if and only if it is finitely supported with respect to the permutation action $\star$.*

PROPOSITION 3.4 ([33]) *Let $(X,\cdot)$ and $(Y,\diamond)$ be invariant sets. Let $f \in Y^X$ and $\pi \in S_A$ be arbitrary elements. Let $\star : S_A \times Y^X \to Y^X$ be the S_A-action on Y^X defined by: $(\pi \star f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. Then $\pi \star f = f$ if and only if for all $x \in X$ we have $f(\pi \cdot x) = \pi \diamond f(x)$.*

Definition 3.6 can be generalized in the following way:

DEFINITION 3.7 *Let X and Y be invariant sets, and let Z be a finitely supported subset of X. A function $f : Z \to Y$ is* finitely supported *if $f \in \wp_{fs}(X \times Y)$.*

The following characterisation can be proved in a manner analogous to that of Proposition 3.4.

PROPOSITION 3.5 *Let $(X,\cdot)$ and $(Y,\diamond)$ be invariant sets, and let Z be a finitely supported subset of X. Let $f : Z \to Y$ be a function. The function f is finitely supported in the sense of Definition 3.7 if and only if there exists a finite set S of atoms such that for all $x \in Z$ and all $\pi \in Fix(S)$ we have $\pi \cdot x \in Z$ and $f(\pi \cdot x) = \pi \diamond f(x)$* [2].

Proof: We assume that f is finitely supported in the sense of Definition 3.7. There exists a finite set S of atoms such that $\pi \star f = f$ for all $\pi \in Fix(S)$, where $\star$ represents the S_A-action on $\wp(X \times Y)$ defined as in Example 3.1(5). Let $x \in Z$ and $\pi \in Fix(S)$ be arbitrary elements. Then there exists an unique $y \in Y$ such that $(x, y) \in f$. Since $\pi \star f = f$ we have $(\pi \cdot x, \pi \diamond y) \in f \subseteq (Z \times Y)$. Thus $\pi \cdot x \in Z$ and $f(\pi \cdot x) = \pi \diamond y = \pi \diamond f(x)$.

Conversely, we assume that there exists a finite set S of atoms such that for all $x \in Z$ and all $\pi \in Fix(S)$ we have $\pi \cdot x \in Z$ and $f(\pi \cdot x) = \pi \diamond f(x)$. We claim that $\pi \star f = f$ for all $\pi \in Fix(S)$. Fix some $\pi \in Fix(S)$, and consider (x, y) an arbitrary element in f. We have $f(x) = y$, and so $(\pi \cdot x, \pi \diamond y) \in f$. However $\pi \triangleright (x, y) = (\pi \cdot x, \pi \diamond y) \in f$, where $\triangleright$ represents the S_A-action on $X \times Y$ defined as in Example 3.1(6). That means $\pi \star f = f$.

Now, if A is a set of atoms (elements with no internal structure), as in [21] we can take a set-theoretic approach and construct a single 'large' S_A-set, i.e. a class $FM(A)$ equipped with an S_A-action, in which all the elements have the finite support property. One benefit is that if a particular construction can be expressed in this language, then the action of permutations is inherited from the ambient universe $FM(A)$ without having to define it explicitly and without having to prove the associated finite support property.

Recall the usual von Neumann cumulative hierarchy of sets:

- $\nu_0 = \emptyset$;
- $\nu_{\alpha+1} = \wp(\nu_\alpha)$;
- $\nu_\lambda = \underset{\alpha \leq \lambda}{\cup} \nu_\alpha$ (λ a limit ordinal).

More generally, we can analogously define a cumulative hierarchy of sets involving atoms from a certain set of atoms U [21]:

[2]The case when Z is equivariant reduces to Proposition 3.4 because the equivariant subsets have properties similar to those of invariant sets.

- $\nu_0(U) = \emptyset$;
- $\nu_{\alpha+1}(U) = U + \wp(\nu_\alpha(U))$;
- $\nu_\lambda(U) = \bigcup_{\alpha\leq\lambda} \nu_\alpha(U)$ (λ a limit ordinal),

where + denotes the disjoint union of sets defined in Example 3.1 (7). Let $\nu(U)$ be the union of all $\nu_\alpha(U)$. The class of sets built on atoms U is $\nu(U)$. We define the notions of S_A-set and finite support property into such a hierarchy by considering U to be the S_A-set A of atoms, and replacing $\wp(-)$ by $\wp_{fs}(-)$ (with the notations of Example 3.1). Thus:

- $FM_0(A) = \emptyset$;
- $FM_{\alpha+1}(A) = A + \wp_{fs}(FM_\alpha(A))$;
- $FM_\lambda(A) = \bigcup_{\alpha\leq\lambda} FM_\alpha(A)$ (λ a limit ordinal).

From Example 3.1 each $FM_\alpha(A)$ is an invariant set. When we consider the union of all $FM_\alpha(A)$ we get one 'large' S_A-set (i.e. an S_A-class) in which every element has finite support. The union of all $FM_\alpha(A)$ is called the *Fraenkel-Mostowski universe* and is denoted by $FM(A)$. Using the names *atm* and *set* for the functions $x \mapsto (0, x)$ and $x \mapsto (1, x)$ (the notations are preserved from Example 3.1) we have that every element x of $FM(A)$ is either of the form $atm(a)$ with $a \in A$, or of the form $set(X)$ where X is a finitely supported set formed at an earlier ordinal stage than x. We call FM-sets the elements of the form $set(X)$, and atoms the elements of the form $atm(a)$.

The S_A-action $\cdot$ on the FM universe $FM(A)$ are defined recursively by:

$$\pi \cdot atm(a) = atm(\pi(a)),\ \pi \cdot set(X) = set(\{\pi \cdot x \,|\, x \in X\}).$$

An element $x \in \nu(A)$ is an FM-set if and only if the following conditions are satisfied:

- y is a FM-set or an atom for all $y \in x$, and
- x has finite support

An FM-set x is not itself closed under the S_A-action on $FM(A)$ unless $supp(x) = \emptyset$. Hence a FM-set is not necessarily an invariant set in the sense of Definition 3.4. However, an FM-set is a finitely supported element of the invariant set $FM(A)$ which additionally has a recursive property of finite support for its elements. Also, an FM-set with empty support is an invariant set.

We provide an axiomatic presentation of the FM set theory. The axioms are the ZFA axioms [20] together with the additional axiom of finite support (axiom 11). We use the symbol "$\Rightarrow$" as "implies", and the symbol "$\Leftrightarrow$" as "if and only if".

DEFINITION 3.8 *The following axioms give a complete characterisation of the Fraenkel-Mostowski set theory:*

1. $\forall x.(\exists y. y \in x) \Rightarrow x \notin A$ (only non-atoms can have elements)

2. $\forall x, y.(x \notin A \, and \; y \notin A \; and \; \forall z.(z \in x \Leftrightarrow z \in y)) \Rightarrow x = y$ (axiom of extensionality)

3. $\forall x, y.\exists z.z = \{x, y\}$ (axiom of pairing)

4. $\forall x.\exists y.y = \{z \,|\, z \subset x\}$ (axiom of powerset)

5. $\forall x.\exists y.y \notin A \, and \, y = \{z \,|\, \exists w.(z \in w \, and \, w \in x)\}$ (axiom of union)

6. $\forall x.\exists y.(y \notin A \, and \, y = \{f(z) \,|\, z \in x\})$, *for each functional formula* $f(z)$ (axiom of replacement)

7. $\forall x.\exists y.(y \notin A \, and \, y = \{z \,|\, z \in x \, and \, p(z)\})$, *for each formula* $p(z)$ (axiom of separation)

8. $(\forall x.(\forall y \in x.p(y)) \Rightarrow p(x)) \Rightarrow \forall x.p(x)$ (induction principle)

9. $\exists x.(\emptyset \in x \, and \, (\forall y.y \in x \Rightarrow y \cup \{y\} \in x))$ (axiom of infinity)

10. *A is not finite.*

11. $\forall x.\exists S \subset A.\, S$ *is finite and* S *supports* x. (finite support property)

It is clear that $\nu(A)$ is a model of ZFA set theory, and $FM(A)$ is a model of FM set theory.

4 Choice Principles in FSM

According to [21], the full axiom of choice fails in FSM. In [21] the authors require the set of atoms to be countable for calculability reasons. However, from the axiomatic construction presented in Section 3, we state that FSM is consistent even when the set of atoms is uncountable and infinite. In this section we prove some stronger results. Regardless of whether if the set of atoms is countable, not only is the full axiom of choice inconsistent in FSM, but many other choice principles presented in Section 2 are also inconsistent in FSM.

Note that the choice principles from Section 2 can be presented in FSM by requiring that all the constructions which appear in their statement are finitely supported. For example:

- **AC** has the form "Given any invariant set X, and any finitely supported family $\mathcal{F}$ of non-empty finitely supported subsets of X, there exists a finitely supported choice function on $\mathcal{F}$".

- **DC** has the form "Let R be a non-empty finitely supported relation on a finitely supported subset Y of an invariant set X having the property that for each $x \in Y$ there exists $y \in Y$ with xRy. Then there exists a finitely supported function $f : \omega \to Y$ such that $f(n)Rf(n+1)$, $\forall n \in \omega$".

- **CC** has the form "Given any invariant set X, and any countable family $\mathcal{F} = (X_n)_n$ of subsets of X such that the mapping $n \mapsto X_n$ is finitely supported, there exists a finitely supported choice function on $\mathcal{F}$".

- **PCC** has the form "Given any invariant set X, and any countable family $\mathcal{F} = (X_n)_n$ of subsets of X such that the mapping $n \mapsto X_n$ is finitely supported, there exists an infinite subset M of $\mathbb{N}$ with the property that there is a finitely supported choice function on $(X_m)_{m \in M}$".

- **AC(fin)** has the form "Given any invariant set X, and any finitely supported family $\mathcal{F}$ of non-empty finite subsets of X, there exists a finitely supported choice function on $\mathcal{F}$".

- **CC(fin)** has the form "Given any invariant set X, and any countable family $\mathcal{F} = (X_n)_n$ of finite subsets of X such that the mapping $n \mapsto X_n$ is finitely supported, there exists a finitely supported choice function on $\mathcal{F}$"

- **PCC(fin)** has the form "Given any invariant set X, and any countable family $\mathcal{F} = (X_n)_n$ of finite subsets of X such that the mapping $n \mapsto X_n$ is finitely supported, there exists an infinite subset M of $\mathbb{N}$ with the property that there is a finitely supported choice function on $(X_m)_{m \in M}$".

- **PIT** has the form: "Every invariant boolean algebra [3] with $0 \neq 1$ has a maximal finitely supported ideal".

- **UFT** has the form "Any finitely supported filter of an invariant boolean algebra can be extended to a finitely supported ultrafilter".

- **KW** has the form "Given any invariant set X, and any finitely supported family $\mathcal{F}$ of non-empty finitely supported subsets of X of cardinality at least 2, there exists a finitely supported function f on $\mathcal{F}$ such that $f(Y)$ is a proper non-empty subset of Y for each $Y \in \mathcal{F}$".

- **RKW** has the form "Given any invariant set X, and $\mathcal{F}$ the family of all non-empty finitely supported subsets of X of cardinality at least 2, there exists a finitely supported function f on $\mathcal{F}$ such that $f(Y)$ is a proper non-empty subset of Y for each $Y \in \mathcal{F}$".

- **OP** has the form "For every invariant set X there exists a finitely supported total order relation on X".

- **OEP** has the form "Every finitely supported partial order relation on an invariant set can be enlarged to a finitely supported total order relation".

- **Fin** has the form "Given any infinite finitely supported subset X of an invariant set, there exists a finitely supported injection from $\mathbb{N}$ to X".

THEOREM 4.1 *In FSM, the following implications remain valid.*

1. ***AC*** $\Rightarrow$ ***DC*** $\Rightarrow$ ***CC*** $\Rightarrow$ ***CC(fin)****;*
2. ***PIT*** $\Leftrightarrow$ ***UFT****;*

[3]An invariant boolean algebra is an invariant set $(L, \cdot)$ endowed with an equivariant lattice order $\sqsubseteq$ and with the additional condition that L is distributive and uniquely complemented.

3. ***AC*** $\Rightarrow$ ***UFT*** $\Rightarrow$ ***OEP*** $\Rightarrow$ ***OP*** $\Rightarrow$ ***AC(fin)*** $\Rightarrow$ ***CC(fin)***;

4. ***Fin*** $\Rightarrow$ ***PCC(fin)***;

5. ***KW*** $\Leftrightarrow$ ***RKW*** $\Rightarrow$ ***OP***.

Proof: Theorem 2.1 is valid in FSM is we are able to rephrase it such that all the objects that appear in its proof are finitely supported. In order to make our point we follow [26]:

Each set S_x used in Theorem 2.12 (1) from [26] to prove that **AC** $\Rightarrow$ **DC** is supported by $supp(x) \cup supp(\rho)$; the family $(S_x)_{x \in X}$ and the mapping $x \mapsto S_x$ are supported by $supp(\rho)$, and the sequence $(x_n)_n$ is supported by $supp(x_0) \cup supp(s)$ where s is the finitely supported choice function on the family S_x. [4]

In order to prove that **DC** $\Rightarrow$ **CC** we restate Theorem 2.12 (2) from [26] in FSM. Let X be an invariant set and $\mathcal{F} = (X_n)_{n>0}$ be a family of subsets of X such that the mapping $n \mapsto X_n$ is finitely supported (i.e. all X_n are supported by the same set). Define $Y_n = \prod_{m \leq n} X_m$ and $Y = \bigcup_{n>0} Y_n$. All Y_n and Y are supported by the finite set which supports the mapping $n \mapsto X_n$. Consider the relation ρ on Y defined by: $(x_1, \dots, x_n)\rho(z_1, \dots, z_m) \Leftrightarrow (m = n+1$ and $x_i = z_i$ for $i = 1, \dots, n)$. The relation ρ is also supported by the finite set which supports the mapping $n \mapsto X_n$ because $supp(n \mapsto X_n)$ supports Y and each permutation is a bijective function. Therefore, according to **DC** there exists a sequence $(y_n)_{n>0}$ in Y such that $y_n \rho y_{n+1}$, and all y_n are supported by the same set S (i.e. the mapping $n \mapsto y_n$ is supported by S). If we assume that $y_1 = (x_1)$, $x_1 \in X_1$, then, according to the definition of ρ we have that y_2 is a pair (x_1, x_2), with $x_2 \in X_2$, y_3 is a 3-tuple (x_1, x_2, x_3), with $x_3 \in X_3$, and so on. Therefore each y_n is an n-tuple of form (y_{n-1}, x_n), $x_n \in X_n$. By induction, since we know that all y_n are supported by the same S, it follows that all x_n are supported by the same S. The family $(x_n)_{n>0}$ is a finitely supported family in $\prod_{n>0} X_n$, and so **CC** is a valid statement in FSM. Note that the assumption that $y_1 = (x_1) \in X_1$ is made for an easy writing. If there exists some $k > 1$ such that $y_1 \in \prod_{m \leq k} X_m$, then, in a similar way (using the definition of ρ), we prove that $\prod_{k \leq m} X_m$ is non-empty. Hence $\prod_{n>0} X_n$ is also non-empty because the cartesian product of a finite family is obviously non-empty.

The proof of Theorem 4.39 from [26] can be reformulated in FSM since each finite subset F of X is finitely supported, and the mappings $F \mapsto X_F$ and $(E, F) \mapsto A_{(E,F)}$ are finitely supported by $supp(F) \cup supp(R)$ and $supp(E) \cup supp(F) \cup supp(R)$, respectively [5]; it follows that **UFT** $\Rightarrow$ **OEP** holds in FSM. According to Theorem 4.39 in [26] we obtain directly that **OEP** $\Rightarrow$ **OP** holds in FSM. Now, if we assume that **OP** holds in FSM, and we consider a finitely supported family $\mathcal{F}$ of non-empty finite subsets of an invariant set U, then $X = \cup \mathcal{F}$ is supported by $supp(\mathcal{F})$. Since there exists a finitely supported linear order on U, it follows that there exists a finitely supported linear order $<$ on X [6], and so each $Y \in \mathcal{F}$ has a least smallest element y_Y. We can define a choice function f on $\mathcal{F}$ by $f(Y) = y_Y$ for all $Y \in \mathcal{F}$. We claim that $supp(\mathcal{F}) \cup supp(<)$ supports f. Let $\pi \in Fix(supp(\mathcal{F}) \cup supp(<))$ and let

[4] The notations are those used in the proof of Theorem 2.12 from [26].

[5] The notations are those used in the proof of Theorem 4.39 from [26].

[6] $<$ is the restriction to X of the linear order on U. If we denote by R the finitely supported linear order on U, then $<$ is supported by $supp(R) \cup supp(X)$

us fix $Y \in \mathcal{F}$. Since $\pi \star Y \in \mathcal{F}$, according to Proposition 3.5 it remains to prove that $f(\pi \star Y) = \pi \cdot y_Y$ (where $\star$ is defined as in Example 3.1 (5)). We know that, because π fixes $supp(\mathcal{F})$ pointwise, there exists some $Z \in \mathcal{F}$ such that $\pi \star Y = Z$. For each $z \in Z$ we obtain an element $y \in Y$ such that $z = \pi \cdot y$. However $y_Y < y$ according to the definition of y_Y. Since π fixes $supp(<)$ pointwise we have $\pi \cdot y_Y < \pi \cdot y = z$ [7]. Moreover, $\pi \cdot y_Y \in (\pi \star Y) = Z$. Since z was arbitrarily chosen from Z we obtain that $\pi \cdot y_Y$ is the smallest element of Z. Therefore, $f(\pi \star Y) = f(Z) = \pi \cdot y_Y = \pi \cdot f(Y)$. It follows that the result **OP** $\Rightarrow$ **AC(fin)** is valid in FSM.

In order to prove Theorem 4.1 (4), we assume that **Fin** is a valid statement. Let $(X_n)_n$ be a sequence of finite subsets of an invariant set such that the mapping $n \mapsto X_n$ is finitely supported (i.e. all X_n are supported by the same S). The set $X = \cup(X_n \times \{n\})$ in Theorem 2.14 (2) from [26] is obviously finitely supported by $supp(n \mapsto X_n)$. Therefore X has a denumerable subset $Y = (y_k)_k$ with the property that the mapping $k \mapsto y_k$ is finitely supported. The set $M = \{n \in \mathbb{N} \,|\, Y \cap (X_n \times \{n\}) \neq \emptyset\}$ in the proof of Theorem 2.14 (2) from [26] is infinite, well defined in FSM, and equivariant because it is a subset of $\mathbb{N}$. For each $m \in M$ define $k_m = min\{k \in \mathbb{N} \,|\, y_k \in (X_m \times \{m\})\}$. Then $y_{k_m} = (x_m, m)$ for a unique $x_m \in X_m$. Fix some $m \in M$. For each $\pi \in supp(k \mapsto y_k)$ we have $\pi \cdot y_{k_m} = (\pi \cdot x_m, \pi \cdot m) = (\pi \cdot x_m, m)$; moreover, we also have $\pi \cdot y_{k_m} = y_{k_m} = (x_m, m)$. That means $supp(k \mapsto y_k)$ supports x_m. Therefore the sequence $(x_m)_{m \in M}$ is finitely supported (and the mapping $m \mapsto x_m$ is finitely supported). It follows that **Fin** $\Rightarrow$ **PCC(fin)** holds in FSM.

In order to prove Theorem 4.1 (5), we restate Theorem 4.40 from [26] in FSM. We assume that **KW** is valid in FSM, and consider an invariant set $(X, \cdot)$. Let $\mathcal{F}$ be the set of all finitely supported subsets Y of X with cardinal greater than or equal to 2. Since permutations are bijective functions, we claim that $\mathcal{F}$ is equivariant. Indeed, for each $\pi \in S_A$ we have that $\pi \star Y$ is finitely supported if Y is finitely supported (see Proposition 3.2), and $\pi \star Y$ has at least 2 elements if Y has at least two elements. The implication **KW** $\Rightarrow$ **RKW** follows immediately by applying **KW** to the finitely supported family $\mathcal{F}$. Now, assume that **RKW** is a valid choice principle, and consider an invariant set $(X, \cdot)$. Let $\mathcal{F}$ be a finitely supported family of finitely supported subsets Y of X with cardinal greater than or equal to 2. Let $\mathcal{F}'$ be the set of all finitely supported subsets Y of X with cardinal greater than or equal to 2. For $\mathcal{F}'$ there exists a finitely supported function f on $\mathcal{F}'$ such that $f(Y)$ is a proper non-empty subset of Y for each $Y \in \mathcal{F}'$. However, $\mathcal{F} \subseteq \mathcal{F}'$. Since the function $g = f|_{\mathcal{F}}$ is supported by $supp(\mathcal{F}) \cup supp(f)$ (see Proposition 3.5), we obtain that g is the required Kinna-Wagner selection function on $\mathcal{F}$.

Now, assume that **RKW** is valid in FSM, and consider an invariant set $(X, \cdot)$. Let $\mathcal{F}$ be the set of all finitely supported subsets Y of X with cardinal greater than or equal to 2. By **RKW** there exists a family $(U_Y)_{Y \in \mathcal{F}}$ of non-empty proper finitely supported subsets U_Y of Y such that the mapping $Y \mapsto U_Y$ is finitely supported. That means $\pi \star U_Y = U_{\pi \star Y}$ for all $\pi \in Fix(supp(Y \mapsto U_Y))$ (see Proposition 3.5). Denote $Y \setminus U_Y$ by V_Y. Obviously, each V_Y is supported by the union between the support of Y and the support of the related U_Y. We claim that $Y \mapsto V_Y$ is supported by $supp(Y \mapsto U_Y)$. Indeed, let $\pi \in Fix(supp(Y \mapsto U_Y))$. It follows that $V_{\pi \star Y} = (\pi \star Y) \setminus U_{\pi \star Y} = (\pi \star Y) \setminus (\pi \star U_Y) = \pi \star (Y \setminus U_Y) = \pi \star V_Y$.

[7] The relation $\pi \cdot y_Y < \pi \cdot y$ makes sense because $\pi \in Fix(supp(\mathcal{F}))$ and $supp(\mathcal{F})$ supports X, and so both $\pi \cdot y_Y$ and $\pi \cdot y$ are elements from X.

Therefore, $\pi \star V_Y = V_{\pi\star Y}$ for all $\pi \in Fix(supp(Y \mapsto U_Y))$, that is, $Y \mapsto V_Y$ is supported by $supp(Y \mapsto U_Y)$. Consider the set Z of all finitely supported linear preorder relations R on X. According to Proposition 3.2, Z is equivariant. For each $R \in Z$ and each $x \in X$ consider the component $[x]_R = \{y \in X \,|\, xRy\, and\, yRx\}$ of x in (X, R). We have that each $[x]_R$ is supported by $supp(x) \cup supp(R)$. Let $\mathcal{K}_R$ be the set of all components $[x]_R$ of (X, R) with at least two elements. Since $\pi \star [x]_R = [\pi \cdot x]_R$ for all $\pi \in Fix(supp(R))$ and each permutation is bijective we have that $\pi \star [x]_R \in \mathcal{K}_R$ for each $\pi \in Fix(supp(R))$ and each $[x]_R \in \mathcal{K}_R$. That means $supp(R)$ supports $\mathcal{K}_R$. Let $\aleph$ be the Hartogs-number of Z, i.e., $\aleph$ is the least upper bound of the set $\{\alpha \,|\alpha$ is an ordinal with $|\alpha| \leq |Z|\}$ [24] . We define, via transfinite recursion, a map $f : \aleph \to Z$ [8] by:

- $f(0) = X \times X$;
- $f(\alpha + 1) = f(\alpha) \setminus \cup\{V_K \times U_K \,|\, K \in \mathcal{K}_{f(\alpha)}\}$;
- $f(\alpha) = \underset{\beta<\alpha}{\cap} f(\beta)$, if α is a limit ordinal.

If $supp(f(\alpha))$ exists, we claim that $f(\alpha+1)$ is supported by the set $supp(f(\alpha)) \cup supp(Y \mapsto U_Y)$. Let $\pi \in Fix(supp(f(\alpha)) \cup supp(Y \mapsto U_Y))$. Let us consider $(v, u) \in \cup\{V_K \times U_K \,|\, K \in \mathcal{K}_{f(\alpha)}\}$. That means there exists $K_0 \in \mathcal{K}_{f(\alpha)}$ such that $(v, u) \in V_{K_0} \times U_{K_0}$. Since $\pi \in Fix(supp(Y \mapsto U_Y))$ we have $(\pi \cdot v, \pi \cdot u) \in V_{\pi\star K_0} \times U_{\pi\star K_0}$. However we proved that $supp(f(\alpha))$ supports $\mathcal{K}_{f(\alpha)}$. Since $\pi \in Fix(supp(f(\alpha)))$ and $K_0 \in \mathcal{K}_{f(\alpha)}$, we have $\pi \star K_0 \in \mathcal{K}_{f(\alpha)}$. Thus $\pi \star (v, u) \in \cup\{V_K \times U_K \,|\, K \in \mathcal{K}_{f(\alpha)}\}$. Now, consider $(v', u') \in f(\alpha + 1)$. Clearly, $\pi \star (v', u') \in f(\alpha)$ because we have $\pi \in Fix(supp(f(\alpha)))$. If $\pi \star (v', u') \in \cup\{V_K \times U_K \,|\, K \in \mathcal{K}_{f(\alpha)}\}$, then we have $(v', u') = \pi^{-1} \star (\pi \star (v', u')) \in \cup\{V_K \times U_K \,|\, K \in \mathcal{K}_{f(\alpha)}\}$ which contradicts the choice of (v', u'). That means $\pi \star (v', u') \in f(\alpha + 1)$.

Clearly, $supp(f(0)) = \emptyset$, $f(1)$ is supported by $supp(Y \mapsto U_Y)$, and so on. Therefore $f(\alpha+1)$ is supported by $supp(Y \mapsto U_Y)$ (which does not depend on α) for each $\alpha \in \aleph$. That means f is finitely supported.

Since $\aleph \not\leq |Z|$, f cannot be injective. Thus there exists some $\alpha \in \aleph$ with $f(\alpha+1) = f(\alpha)$. For this α, $\mathcal{K}_{f(\alpha)}$ must be empty, i.e., $f(\alpha)$ is the required finitely supported linear order on X. That means **RKW** $\Rightarrow$ **OP** is a valid result in FSM.

The remaining results in Theorem 4.1 can be translated in FSM in a similar way.

Therefore the relationship between the choice principles presented in Theorem 2.1 is preserved in FSM, with the exception of **CC** $\Rightarrow$ **Fin** and **PCC** $\Rightarrow$ **CC**[9], and with the clarification that in FSM the choice principles are presented in terms of finitely supported objects.

THEOREM 4.2 *The choice principle **Fin** is inconsistent in FSM.*

Proof: Let us assume that **Fin** is valid in FSM. Therefore we can find a finitely supported injection $f : \mathbb{N} \to A$. Let us consider $m, n \in \mathbb{N}$ such that $m \neq n$ and $f(m), f(n) \notin supp(f)$.

[8]Such a map can be correctly defined in FSM because Z is equivariant and, hence, it has the same properties as an invariant set.

[9]These implications are not important in FSM. If we look at Theorem 4.2 and Theorem 4.6, the choice priciples **Fin**, **PCC** and **CC** fail in FSM.

Hence $(f(m)\,f(n)) \star f = f$. Let us denote $(f(m)\,f(n))$ by π. Since the S_A-action $\cdot$ on $\mathbb{N}$ is defined as in Example 3.1 (4), according to Proposition 3.3 we have $f(m) = (\pi \star f)(m) = \pi(f(m)) = f(n)$ which contradicts the injectivity of f. Thus **Fin** is inconsistent in FSM.

THEOREM 4.3 *The choice principle* ***AC(fin)*** *is inconsistent in FSM.*

Proof: Let us assume that **AC(fin)** is valid in FSM. We consider the set $P := \wp_2(A) = \{X|X \subset A,\, X\, \textit{finite},\, |X| = \textit{2}\}$. That means P is the set of *all* pairs of elements from A. Since P is equivariant (i.e. empty supported) our construction makes sense in FSM. We asuume that **AC(fin)** is valid which means there exists a finitely supported choice function f on P. Let $Y = \{a_1, \dots, a_n, b_1, \dots, b_n\}$ be a finite set supporting f, where each $\{a_i, b_i\} \in P$. Since P is infinite we may select a pair $\{c, d\} = X$ from P such that c and d are different from all a_i, b_i. Let $\pi \in S_A$ be a permutation which fixes each a_i and b_i and interchanges c and d. Then π fixes f. Since f is a choice function on P, and $X \in P$, we have $f(X) \in X$, that is, $f(X) = c$ or $f(X) = d$. Since π interchanges c and d we have $\pi(f(X)) \neq f(X)$. However $\pi(X) = X$ and, since π fixes Y pointwise (and, hence, π fixes f), we have $\pi(f(X)) = f(\pi(X)) = f(X)$ according to Proposition 3.5 and Example 3.1(1). Thus we get a contradiction, and **AC(fin)** cannot be valid.

Corollary 4.4 *The choice principles* ***PIT****,* ***UFT****,* ***OP*** *and* ***OEP*** *are inconsistent in FSM.*

Proof: The result follows from Theorem 4.1(2,3) and Theorem 4.3.

Corollary 4.5 *The choice principles* ***KW*** *and* ***RKW*** *are inconsistent in FSM.*

Proof: The result follows from Theorem 4.1(5) and Corollary 4.4.

THEOREM 4.6 *The choice principles* ***CC*** *and* ***PCC*** *are inconsistent in FSM.*

Proof: Let us assume that **CC** is valid in FSM. We consider the countable family $(X_n)_n$ where X_n is the set of all injective n-tuples from A. Since A is infinite it follows that each X_n is non-empty. In FSM, each X_n is equivariant because A is an invariant set and each permutation is a bijective function. Therefore the family $(X_n)_n$ is equivariant and the mapping $n \mapsto X_n$ is also equivariant.

If we assume that **CC** is valid, then according to the formulation of **CC** in FSM, there exists a finitely supported choice function f on $(X_n)_n$. Let $f(X_n) = y_n$ with each $y_n \in X_n$. Let $\pi \in Fix(supp(f))$. According to Proposition 3.5, and because each element X_n is equivariant according to its definition, we obtain that $\pi \cdot y_n = \pi \cdot f(X_n) = f(\pi \cdot X_n) = f(X_n) = y_n$. Therefore each element y_n is supported by $supp(f)$. However, since each y_n is a finite tuple of atoms we have $supp(y_n) = y_n$, $\forall n \in \mathbb{N}$. Since $supp(y_n) \subseteq supp(f)$, $\forall n \in \mathbb{N}$, we obtain $y_n \subseteq supp(f)$, $\forall n \in \mathbb{N}$. Since each y_n has exactly n elements, this contradicts the finiteness of $supp(f)$.

If we assume that **PCC** is valid, then according to the formulation of **PCC** in FSM, there exists an infinite subset M of $\mathbb{N}$ and a finitely supported choice function g on $(X_m)_{m \in M}$. Let $g(X_m) = y_m$ with each $y_m \in X_m$. As in the paragraph above we obtain $y_m \subseteq supp(g)$ for all $m \in M$. Since y_m has exactly m elements for each $m \in M$, and since M is infinite we contradict the finiteness of $supp(g)$.

Corollary 4.7 *The choice principles **AC** and **DC** are inconsistent in FSM.*

Proof: The result follows from Theorem 4.1(1) and Theorem 4.6.

Remark 4.8 *According to [15], the following implication holds in the ZF framework:* **AC(fin)** implies that "every infinite set X has an infinite subset Y such that $X \setminus Y$ is also infinite". *A similar result hold for **Fin** according to [29] and for **KW** according to [26]. However we cannot directly conclude that such an implication holds in (can be reformulated in) FSM where only finitely supported objects are allowed. We cannot say (without a proof which is consistent in FSM) that the following statement is valid: " 'Given any invariant set X, and any finitely supported family $\mathcal{F}$ of non-empty finite subsets of X, there exists a finitely supported choice function on $\mathcal{F}$ (i.e, **AC(Fin)** in FSM)' implies 'Every infinite invariant set X has an infinite finitely supported subset Y such that $X \setminus Y$ is also finitely supported and infinite' "* [10]*. Therefore we cannot directly conclude that **AC(fin)** is false in FSM just because the statement "Every infinite set X has an infinite subset Y such that $X \setminus Y$ is also infinite" is false in FSM. Such a result requires a separate proof reformulated in terms of finitely supported objects. An example of how a ZF theorem is translated in the framework of invariant sets is Theorem 4.1.*

In fact, Remark 4.8 states that we cannot prove an FSM result only by employing a ZF result (without an additional proof made according to the finite support requirement). There exist a lot of results which are valid in the ZF framework, but fail in FSM. This is because given an invariant set there could exist some subsets of that set which fail to be finitely supported. For example the simultaneously infinite and coinfinite subsets of A are not finitely supported. An example of a mathematical result that fails in FSM is the Stone representation theorem for Boolean lattices (claiming that every Boolean lattice is isomorphic to the dual algebra of its associated Stone space). According to Subsection 5.2 from [31], Stone duality fails in the framework of invariant sets. Other results which fail in FSM, such as determinization of finite automata, or equivalence of two-way and one-way finite automata, are presented in [11]. More such examples are presented in Section 3 from [1].

Remark 4.8 justifies the non-triviality of Theorem 4.1, Theorem 4.2, Theorem 4.3, and Theorem 4.6

According to the results in this section, the choice principles **AC**, **DC**, **CC**, **PCC**, **PIT**, **UFT**, **OP**, **KW**, **RKW** and **OEP** presented in Section 2 (and reformulated in terms of finitely supported objects) are inconsistent in FSM.

Remark 4.9 *Note that the inconsistence of **UFT** in FSM can also be proved without using Theorem 4.1. From Proposition 5.2.2 in [31]* [11]*, there exists an invariant Boolean algebra having a finitely supported filter that cannot be extended to a finitely-supported ultrafilter. Therefore **UFT** fails in the framework of invariant sets. However we believe that the proof provided in [31] is more difficult than the proof presented in Theorem 4.1.*

[10] Note that the previous implication obviously follows in FSM. However this thing happens because **AC(Fin)** is false in FSM (Theorem 4.3), and a false statement implies everything. In order to prove that the related implication is valid we need first to employ Theorem 4.3.

[11] The proof of this result remains valid even when the set of atoms is not countable

Note that, according to the definition of an FM-set, the previous choice principles can as well be reformulated in terms of FM-sets by informally replacing "finitely supported subset of an invariant set" with "FM-set". For example, **AC** can be reformulated in the form: "given any finitely supported family $\mathcal{F}$ of non-empty FM-sets, there exists a finitely supported choice function on $\mathcal{F}$", and so on. The inconsistence of various choice principles in the Fraenkel-Mostowski cumulative universe can be proved in a similar way as we proved the inconsistence of the related choice principles in FSM. It is obvious that the choice principles presented in Section 2 are inconsistent with the axioms of the FM set theory.

5 Properties of Invariant Sets Obtained by Comparing Choice Principles

Looking back, Theorem 4.1 seems to lose its importance. All the related choice principles compared in Theorem 4.1 are inconsistent in FSM. The reader might ask why we chose to present the more difficult Theorem 4.1 instead of searching for a simpler direct method to prove the inconsistence of each choice principle in FSM. The answer is represented by the remark that the proof of Theorem 4.1 also provides some other interesting choice and order properties of invariant sets.

From the proof of Theorem 4.1(5) we obtain:

THEOREM 5.1 *Let $(X, \cdot)$ be an invariant set. Let $\mathcal{F}$ be the set of all finitely supported subsets Y of X with cardinal greater than or equal to 2. If there exists a family $(U_Y)_{Y\in\mathcal{F}}$ of non-empty proper finitely supported subsets U_Y of Y such that the mapping $Y \mapsto U_Y$ is finitely supported, then there exists a finitely supported linear order on X.*

From the proof of Theorem 4.1(3) we obtain:

THEOREM 5.2 *Let $(U, \cdot)$ be an invariant set. Let us consider a finitely supported family $\mathcal{F}$ of non-empty finite subsets of U. If there exists a finitely supported linear order on U, then there exists a finitely supported function defined on $\mathcal{F}$ with the property that $f(Y) \in Y$ for all $Y \in \mathcal{F}$.*

From the proof of Theorem 4.1(4) we obtain:

THEOREM 5.3 *Let $(X_n)_n$ be a sequence of finite subsets of an invariant set such that the mapping $n \mapsto X_n$ is finitely supported (i.e. all X_n are supported by the same S). Let $X = \cup(X_n \times \{n\})$. If there exists a finitely supported injection $i : \mathbb{N} \to X$, then there exists an infinite subset M of $\mathbb{N}$ and a sequence $(x_m)_{m\in M}$ with the properties that $x_m \in X_m$, $\forall m \in M$ and the mapping $m \mapsto x_m$ is finitely supported.*

6 Conclusion

The Finitely Supported Mathematics (FSM) generalizes the classical Zermelo-Fraenkel (ZF) mathematics, and represents an appropriate framework to work with (infinite) structures in terms of finitely supported objects. FSM is a mathematics which is consistent with the axioms of the Fraenkel-Mostowski set theory. Rather than using a non-standard set theory, we could alternatively work with invariant sets, which are defined within ZF framework as usual sets endowed with some group actions satisfying a finite support requirement. Actually, FSM represents the ZF set theory rephrased in terms of finitely supported objects. The theory of nominal sets [33], rephrased for possible non-countable sets of atoms, could be considered as a tool for studying FSM.

The consistence of the several choice principles with the axioms of the ZF set theory was studied in depth during the last century. Our goal is to study the consistence of various choice principles internally in FSM. It is well known that the full axiom of choice is inconsistent in FSM. However, there exist no results regarding the consistency of various weaker forms of choice in FSM. The choice principles presented in Section 2 can be reformulated in FSM by considering that each object appearing in their statement is finitely supported. The relationship between the related choice principles presented in Theorem 2.1 is preserved in FSM according to Theorem 4.1. According to Theorem 4.1, Theorem 4.2, Theorem 4.3, Theorem 4.6, Corollary 4.4, Corollary 4.5, and Corollary 4.7, the choice principles **AC**, **DC**, **CC**, **PCC**, **AC(fin)**, **Fin**, **PIT**, **UFT**, **OP**, **KW**, **RKW** and **OEP** presented in Section 2 are inconsistent in FSM. Thus, all the results from functional analysis, real analysis, topology, the theory of differential equations, the theory of algebraic structures, measure theory, first-order logic, which require a choice principle in their proof fail in the FSM. Since the theory of invariant sets makes sense even if the set of atoms is not countable, the inconsistence results presented in this paper do not overlap on some related properties in the basic or in the second Fraenkel modes of the ZFA set theory (which are defined using countable sets of atoms) [28]. Also, the results in this paper do not follow immediately from [33] because in [33] the nominal sets are defined over countable sets of atoms whilst we defined invariant sets over possible non-countable sets of atoms; in the viewpoint from [33] (i.e. if the set of atoms is countable) the inconsistency of the countable choice principles would be trivial. Since no information about the countability of the set of atoms is available in a general theory of invariant sets, the consistency of **CC(fin)** in FSM remains an open problem. In the particular case when the set A of atoms is countable, the set P used in the proof of Theorem 4.3 is countable. Therefore, in this case, the choice principle **CC(fin)** is inconsistent in FSM.

Note that various relationships between several forms of choice hold in the ZF framework. However, nobody guarantees that such results remain valid in FSM. When we work in FSM we cannot employ a ZF theorem in order to prove that a certain choice principle is valid or not. Therefore all the FSM relationship results between various choice principles have to be independently proved according to the finite support requirement. This is the reason why the results in Section 4 are not trivial (see, especially, Theorem 4.1 and Remark 4.8). Theorem 4.1 also comes with another benefit. According to its proof we can provide some new choice and order properties of invariant sets. Several such properties are presented in Section 5.

References

[1] Alexandru, A., Ciobanu, G.: Defining Finitely Supported Mathematics over sets with atoms. In: 4th International Workshop on Algebraic, Logical, and Algorithmic Methods

of System Modeling, Specification and Verification–ICTERI, invited talk (2015)

[2] Alexandru, A., Ciobanu, G.: Nominal groups and their homomorphism theorems. Fundamenta Informaticae. 131(3-4), 279–298 (2014)

[3] Alexandru, A., Ciobanu, G.: On the development of the Fraenkel-Mostowski set theory. Bulletin Inst. Politehnic Iasi. LX, 77–91 (2014)

[4] Alexandru, A., Ciobanu, G.: A nominal approach for fusion calculus. Romanian Journal of Information Science and Technology. 17, (2014)

[5] Alexandru, A., Ciobanu, G.: Mathematics of multisets in the Fraenkel-Mostowski framework. Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie. 58/106 (1), 3–18 (2015)

[6] Banach, S., Tarski, A.: Sur la décomposition des ensembles de points en parties respectivement congruentes. Fundamenta Mathematicae. 6 , 244–277 (1924)

[7] Barwise, J.: Admissible Sets and Structures: An Approach to Definability Theory, Perspectives in Mathematical Logic. Vol.7, Springer (1975)

[8] Bojanczyk, M.: Fraenkel-Mostowski sets with non-homogeneous atoms. Lecture Notes in Computer Science. 7550, 1–5, Springer (2012)

[9] Bojanczyk, M.: Nominal monoids. Theory of Computing Systems. 53, 194–222 (2013)

[10] Bojanczyk, M., Braud L., Klin, B., Lasota, S.: Towards nominal computation. In: 39th ACM POPL, pp. 401–412 (2012)

[11] Bojanczyk, M., Klin, B., Lasota, S.: Automata with group actions. In: 26th Symposium on Logic in Computer Science, pp. 355–364. IEEE Press (2011)

[12] Bojanczyk, M., Klin, B., Lasota, S., Torunczyk, S.: Turing machines with atoms. In: 28th Symposium on Logic in Computer Science, pp.183–192. IEEE Press (2013)

[13] Bojanczyk, M., Lasota, S.: A Machine-independent characterization of timed languages. In: 39th ICALP, 92–103 (2012)

[14] Bojanczyk, M., Torunczyk, S.: Imperative programming in sets with atoms. In: FSTTCS. LIPIcs vol.18, pp. 4–15 (2012)

[15] Brunner, N.: Amorphe Potenzen kompacter Räume. Archiv für mathematische Logik und Grundlagenforschung. 24, 119–135 (1984)

[16] Cohen, P.J.: The Independence of the Axiom of Choice, Mimeographed (1963)

[17] Cohen, P.J.: The independence of the continuum hypothesis, I, II. Proc. Nat. Acad. Sci. USA. 50, 1143–1148 (1963) and 51, 105–110 (1964)

[18] Fraenkel, A.: Zu den grundlagen der Cantor-Zermeloschen mengenlehre. Mathematische Annalen. 86, 230–237 (1922)

[19] Gabbay, M.J.: The pi-calculus in FM. Thirty Five Years of Automating Mathematics, Kluwer Applied Logic. 28, pp. 247–269 (2003)

[20] Gabbay, M. J.: A study of substitution, using nominal techniques and Fraenkel-Mostowski sets. Theoretical Computer Science, 410, 1159–1189 (2009)

[21] Gabbay, M.J., Pitts, A.M.: A new approach to abstract syntax with variable binding. Formal Aspects of Computing. 13, 341–363 (2001)

[22] Gödel, K.: The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis with the Axioms of Set Theory. Annals of Mathematics Studies. Princeton University Press (1940)

[23] Goodman, N., Myhill, J.: Choice implies excluded middle. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik. 24, 461–461 (1978)

[24] Hartogs, F.: Über das Problem der Wohlordnung, Mathematische Annalen. 76, 438–443 (1915)

[25] Henkin, L.: Metamathematical theorems equivalent to the prime ideal theorem for Boolean algebras. Bulletin of the American Mathematical Society. 60, 378-388 (1954)

[26] Herrlich, H. Axiom of Choice. Lecture Notes in Mathematics. Springer (2006)

[27] Howard, P., Rubin, J.E.: Consequences of the Axiom of Choice. Mathematical Surveys and Monographs vol.59. American Mathematical Society (1998)

[28] Jech, T. J.: The Axiom of Choice. Studies in Logic and the Foundations of Mathematics. North-Holland (1973)

[29] Levy, A.: The independence of various definitions of finite. Fundamenta Mathematicae. 46, 1–13 (1958)

[30] Lindenbaum, A., Mostowski, A.: Uber die unabhangigkeit des auswahlsaxioms und einiger seiner folgerungen. Comptes Rendus des Seances de la Societe des Sciences et des Lettres de Varsovie. 31, 27–32 (1938)

[31] Petrisan, D.: Investigations into Algebra and Topology over Nominal Sets. PhD Thesis, University of Leicester (2011)

[32] Pitts. A.M.: Alpha-structural recursion and induction. Journal of the ACM. 53, 459–506 (2006)

[33] Pitts, A.M.: Nominal Sets Names and Symmetry in Computer Science. Cambridge University Press (2013)

[34] Rotmann, J.: An Introduction to Homological Algebra. Springer, Berlin (2009)

[35] Rubin, H., Rubin, J.E. Equivalents of the Axiom of Choice. Studies in Logic. North-Holland, Amsterdam (1985)

[36] Shinwell, M.R.: The Fresh Approach: Functional Programming with Names and Binders. PhD Thesis, University of Cambridge (2005)

[37] Turner, D.: Nominal Domain Theory for Concurrency. Technical Report no.751, University of Cambridge (2009)

[38] Urban, C.: Nominal techniques in Isabelle/HOL. Journal of Automated Reasoning. 40, 327–356 (2008)

[39] Zermelo, E.: Beweis, daβ jede Menge wohlgeordnet werden kann. Mathematische Annalen. 59, 514–516 (1904)