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Continuation Semantics for Concurrency

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Continuation Semantics for Concurrency

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ABSTRACT

This paper presents a continuation semantics satisfying the basic laws of concurrent systems. This semantics is illustrated for a simple CSP-like language extended with communication on multiple channels and synchronization based on join patterns, which also provides operators for nondeterministic choice, sequential and parallel composition. For the language under investigation we present a denotational semantics. Then we prove that the semantic operators designed with continuations obey the concurrency laws such as the associativity or the commutativity of parallel composition. The significance of the results is given mainly by the flexibility provided by the continuations technique which can thus be used to describe concurrent behaviour.

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1 Introduction

The paper deals with fundamental aspects of systems exhibiting concurrency, namely systems consisting of several computing processes interacting among each other. By "process" we understand the behaviour of a (software) system, the execution of a program. Since the concurrent systems are usually complex, it is useful to have the possibility of describing, analyzing and reasoning about the concurrent systems in a precise way. Generally the interactions among processes cannot be predicted, and so the concurrent computation involves nondeterminism. The traditional view of computation as a function from input to output cannot deal properly with nondeterminism; the models of concurrency are provided by *process calculi* like ACP [5], CCS [10] and CSP [9]. These process calculi work with terms, and use an operational semantics to describe the computational dynamics. They are also called *process algebras* because they use algebraic operators and equations among terms to describe the system behaviour.

In general, starting out from a given set of atomic actions, the basic operators are used to compose the actions into more complicated processes. The basic operators are + (denoting alternative composition), ; (denoting sequential composition) and $\|$ (denoting parallel composition). The following basic laws express the general properties of the operators involved in a concurrent system. These are static laws because they do describe the action executions explicitly.

$$
\begin{array}{rcl}
x+y &=& y+x(commutativity\ of\ alternative\ composition)\\
x+(y+z) &=& (x+y)+z(associativity\ of\ alternative\ composition)\\
x+x &=& x(idempotency\ of\ alternative\ composition)\\
(x+y);z &=& x;z+y;z\ (left\ distributivity\ of\ +\ over\ ;)\\
x;(y+z) &=& x;y+x;z\ (right\ distributivity\ of\ +\ over\ ;)\\
(x;y);z &=& x;(y;z)\quad (associativity\ of\ sequential\ composition)\\
x\|y &=& y\|x\quad (commutativity\ of\ parallel\ composition)\\
(x\|y)\|z &=& x\|(y\|z)(associativity\ of\ parallel\ composition)
\end{array}
$$

The operator ; binds more tightly than +.

The axiom for the left distributivity of + over ; is not always included in concurrency theories. When this axiom is present one gets a *trace* or *linear-time* semantics where the nondeterministic choice points are not recorded. Without this axiom, one gest a *branching time* model that can provide better flexibility in the modeling of process synchronization.[1] When considering an approach to concurrency theory, parallel composition is the central operator, and so the importance of its proper treatment in a continuation semantics for concurrency.

[1]In this paper we use the notions of *linear time* and *branching time* semantics in the context of denotational semantics. In this context, an element of a linear time domain is a collection of execution traces. An element of a branching time domain is a tree-like structure. For a discussion of linear time versus branching time semantics the reader may consult [2] or the monograph [3].

The denotational semantics of concurrent systems uses the observations of a process (e.g., traces, failures). The meaning of a process is given by a set of possible observations. These denotations are compositional and provide an operational intuition. However the denotational semantics are more abstract than the operational semantics, and are not close to any specific implementation for a concurrent system or language. In this paper we present a *continuation semantics for concurrency* (CSC) which satisfies the static laws of concurrent systems, and which can provide flexibility in designing concurrent languages.

Continuations are known in denotational semantics for the flexibility they provide as a language design tool. Traditional continuations [12] can be used to model a variety of advanced control concepts, including non-local exists, coroutines and even multitasking [16]. However, according to [8], the traditional continuations do not work well-enough in the presence of concurrency. The CSC technique [13] is a general tool for representing control in concurrent systems. A CSC continuation is a structure of computations (partially evaluated denotations) that can be evaluated in parallel.

The structure of CSC continuations is representative for the control concepts of the (concurrent) language under study. In the case of a simple sequential language a continuation is a stack of computations. For handling parallel composition, CSC continuations can be structured as multisets of computations. Communication and synchronization information can also be encoded in continuations. Thanks to the use of continuations, we do not need a branching time domain to model communication and synchronization behavior. The final yield of the denotational semantics is an element of a linear-time domain, i.e. a simple collection of execution traces. The CSC technique provides a "pure" continuation-based approach to communication and concurrency, in which all control concepts are modeled as operations manipulating continuations. In the CSC approach the heavy part of the computation is moved from the powerdomains [11] to the domains of continuations.

In this paper we consider a simple imperative concurrent language MCC extended with communication on multiple channels in the style of Join calculus [7]. MCC is an abbreviation for *Multiple Channels Communication*. The language MCC was first investigated in [14] (although in [14] it was named L_J). MCC provides operators for sequential composition, parallel composition and nondeterministic choice. It also provides two primitives for concurrent interaction on multiple channels: $c!e$, and $c_1?v_1\&\cdots\&c_n?v_n$ which we call a *join pattern*. Synchronized execution of $n+1$ actions $c_1!e_1,\cdots,c_n!e_n$ and $c_1?v_1\&\cdots\&c_n?v_n$ occurring in parallel processes, results in the transmission of the value of each expression e_i along the channel c_i from the process executing the $c_i!e_i$ statement to the process executing the $c_1?v_1\&\cdots\&c_n?v_n$ statement. The latter assigns the n received values to the variables $v_1,\cdots,v_n$: the value of each expression e_i is transmitted along the channel c_i and assigned to the corresponding variable v_i. The whole interaction behaves like a distributed multi-assignment. When $n=1$ the interaction is a simple CSP-like point-to-point communication [9].

We present a denotational model designed with CSC for MCC. The denotational model is built within the mathematical framework of 1-bounded complete metric spaces [3]. We use the general theory developed in [1] as the domain of denotations is defined as solution of an equation in where the domain variable occurs in the left-hand side of a function space construction. We prove that the semantic operators - that are designed with CSC continuations - obey the laws that are usually given in the algebraic theories of concurrency, such as the associativity and the commutativity of parallel composition.

Some of the properties are simple. For example, the semantics of nondeterministic choice is given by the standard union operator, which is associative, commutative and idempotent. Other properties, e.g. the associativity of sequential and parallel execution, require more complex proofs. The basic idea of each of the main proofs is to show that the property under consideration is preserved by the computation steps (it is an invariant of the computation). In metric semantics it is customary to attach a $\frac{1}{2}\cdot$ -contracting factor to each computation step. In each case we obtain a relation of the kind $\epsilon \leq \frac{1}{2}\cdot\epsilon$ where ϵ is the distance between two behaviorally equivalent continuations. Therefore $\epsilon = 0$ and the desired property follows. This proof technique is general and can be applied to every language designed with CSC. Only the structure of continuations and the computation invariants (which are language specific and are related to the structure of continuations) need to be adapted.

2 Preliminaries

The notation $(x \in)X$ introduces the set X with typical element x ranging over X. For any set X we denote by $|X|$ the *cardinal number* of X. Let $f \in X \rightarrow Y$ be a function. The function $[\, f \mid x \mapsto y\,] : X \rightarrow Y$, is defined (for $x, x' \in X, y \in Y$) by: $[\, f \mid x \mapsto y\,](x') =$ $\mathsf{if}\ x'{=}x\ \mathsf{then}\ y\ \mathsf{else}\ f(x')$. We also use the notation $[\, f \mid x_1 \mapsto y_1 \mid \cdots \mid x_n \mapsto y_n\,]$ as an abbreviation for $[\,[\, f \mid x_1 \mapsto y_1\,] \cdots \mid x_n \mapsto y_n\,]$. If $f : X \rightarrow X$ and $f(x) = x$ we call x a *fixed point* of f. When this fixed point is unique (see Theorem 2.1) we write $x = fix(f)$.

The study presented in this paper takes place in the mathematical framework of *1-bounded complete metric spaces*. We work with the following notions which we assume to be known: *metric* and *ultrametric* spaces, *isometry* (distance preserving bijection between metric spaces) denoted by '$\cong$'), and *complete* metric space. For further explanations the reader may consult the monograph [3].

A simple example of ultrametric space can be obtained by endowing an arbitrary nonempty set $(x, y \in)X$ with the metric $d : X \times X \rightarrow [0, 1]$ defined by $d(x, y) = \mathsf{if}\ x = y\ \mathsf{then}\ 0\ \mathsf{else}\ 1$. d is the so-called *discrete* metric on X. (X, d) is a complete ultrametric space.

Let $(a \in)A$ be a nonempty set. We denote by A^* the set of all finite sequences over A, and we denote by A^ω the set of all infinite sequences over A. We obtain another example of ultrametric space by endowing the set $(x, y \in)A^\infty = A^* \cup A^\omega$ with the metric $d : A^\infty \times A^\infty \rightarrow$ $[0, 1]$, given by: $d(x, y) = 2^{-\sup\{\, n \mid x[n] = y[n]\,\}}$, where $x[n]$ denotes the prefix of x of length n if $length(x) \geq n$, and x otherwise. By convention, $2^{-\infty} = 0$. Such a d is a *Baire* metric, and (A^∞, d) is a complete ultrametric space.

We recall that if (X, d_X), (Y, d_Y) are metric spaces, a function $f{:}X \rightarrow Y$ is a *contraction* if $\exists c \in \mathbb{R},\ 0 \leq c < 1, \forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) {\leq} c{\cdot}d_X(x_1, x_2)$. When $c = 1$ the function f is called *non-expansive*. In what follows we denote the set of all nonexpansive functions from X to Y by $X \xrightarrow{1} Y$. The following Theorem is at the core of metric semantics.

THEOREM 2.1 (Banach) *Let (X, d_X) be a complete metric space. Each contraction $f : X \to X$ has a* unique *fixed point.*

DEFINITION 2.1 *Let $(X, d_X), (Y, d_Y)$ be (ultra) metric spaces with $d_X, d_Y \leq 1$. We define the following metrics for $x \in X$, $f \in X \to Y$ (the function space), $(x, y) \in X \times Y$ (the Cartesian product), $u, v \in X + Y$ (the disjoint union of X and Y), and $U, V \in \mathcal{P}(X)$ (the power set of X):*

(a) $d_{\frac{1}{2}\cdot X} : X \times X \to [0, 1] \quad d_{\frac{1}{2}\cdot X}(x_1, x_2) = \frac{1}{2} \cdot\ d_X(x_1, x_2)$

(b) $d_{X \to Y} : (X \to Y) \times (X \to Y) \to [0, 1]$

$$d_{X \to Y}(f_1, f_2) = sup_{x \in X} d_Y(f_1(x), f_2(x))$$

(c) $d_{X \times Y} : (X \times Y) \times (X \times Y) \to [0, 1]$

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

(d) $d_{X+Y} : (X + Y) \times (X + Y) \to [0, 1]$

$d_{X+Y}(u, v) =$

if $(u, v \in X)$ then $d_X(u, v)$

else if $(u, v \in Y)$ then $d_Y(u, v)$ else 1

(The disjoint union can be defined by: $X + Y = (\{1\} \times X) \cup (\{2\} \times Y)$)

(e) $d_H : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, 1]$*:*

$$d_H(U, V) = max\{sup_{u \in U} d(u, V), sup_{v \in V} d(v, U)\}$$

where $d(u,W) = inf_{w \in W} d(u, w)$ *and by convention* $sup\emptyset = 0$, $inf\emptyset = 1$
(d_H is the Hausdorff *distance).*

We use the abbreviations $\mathcal{P}_{nco}(\cdot)$, $\mathcal{P}_{fin}(\cdot)$ and $\mathcal{P}_{nfin}(\cdot)$, to denote the power sets of *non-empty and compact*, *finite* and *non-empty and finite* subsets of '$\cdot$', respectively. In general, the constructs $\mathcal{P}_{fin}(\cdot)$ and $\mathcal{P}_{nfin}(\cdot)$ do not give rise to complete spaces. In our study, we only use the power set of finite subsets to create structures that we endow with the discrete metric; any set endowed with the discrete metric is trivially a complete ultrametric space. Also, we often suppress the metrics part in domain definitions, and write, e.g., $\frac{1}{2} \cdot X$ instead of $(X, d_{\frac{1}{2}\cdot X})$.

Remark 1 *Let $(X, d_X), (Y, d_Y), d_{\frac{1}{2}\cdot X}$, $d_{X \to Y}, d_{X \times Y}$, d_{X+Y} and d_H be as in definition 2.1. In case d_X, d_Y are ultrametrics, so are $d_{\frac{1}{2}\cdot X}, d_{X \to Y}, d_{X \times Y}, d_{X+Y}$ and d_H. Moreover, if $(X, d_X), (Y, d_Y)$ are complete then $\frac{1}{2} \cdot X$, $X \to Y$, $X \xrightarrow{1} Y$, $X \times Y$,$X + Y$, and $\mathcal{P}_{\mathrm{nco}}(X)$ (with the metrics defined above) are also complete metric spaces [3].*

3 Syntax and Continuation Structure for MCC

The syntax of MCC is given in BNF in 3.1. The basic components are a set $(v \in) Var$ of *variables*, a set $(e \in) Exp$ of *expressions*, a set $(c \in) Ch$ of *communication channels* and a set $(x \in) Pvar$ of *procedure variables*. Exp is a class of numeric expressions without side effects that evaluate to integer values $z \in \mathbb{Z}$. We assume that the evaluation of an expression always terminates.

DEFINITION 3.1 *(Syntax of MCC)*

(a) *(Join patterns)* $\quad j(\in J) ::= c?v \mid j \,\&\, j$

For a program to be valid the channels $c_1, \cdots, c_n$ *and the variables* $v_1, \cdots, v_n$ *in a join pattern* $j = (c_1?v_1 \,\&\, \cdots \,\&\, c_n?v_n)$ *must be pairwise distinct.*

(b) *(Statements)* $\quad s(\in Stat) ::= a \mid x \mid s + s \mid s; s \mid s \| s$

where $\quad a ::= \mathsf{skip} \mid v := e \mid c!e \mid j$

(c) *(Guarded statements)* $\quad g(\in GStat) ::= a \mid g + g \mid g; s \mid g \| g$

(d) *(Declarations)* $(D \in) Decl = PVar \to GStat$

(e) *(Programs)* $(\rho \in) MCC = Decl \times Stat$

The language MCC provides assignment $(v := e)$, recursion, sequential composition $(s; s)$, nondeterministic choice $(s + s)$, parallel composition $(s \| s)$ and the communication mechanism that was explained informally in the introduction. Following a standard approach, we assume the meanings of expressions is given by a valuation $\mathcal{E}[\![\cdot]\!] : Exp \to \Sigma \to \mathbb{Z}$, where $(\sigma \in) \Sigma = Var \to \mathbb{Z}$ is a set of *states*.

We employ an approach to recursion based on *declarations* and *guarded statements* as in [3]. In a guarded statement each recursive call is preceded by at least one elementary action, which guarantees the fact that the semantic operators are contracting functions in the present metric setting. For the sake of brevity, in what follows we assume a fixed declaration $D \in Decl$, and all considerations in any given argument refer to this fixed D.

For inductive proofs we introduce a complexity measure that decreases upon recursive calls. A function such as c is well-defined due to our restriction to guarded recursion (Definition 3.1(c), 3.1(d)).

DEFINITION 3.2 *(Complexity measure) The function* $c : Stat \to \mathbb{N}$ *is given by*

$$\begin{array}{rcl} c(a) & = & 1 \\ c(s_1; s_2) & = & 1 + c(s_1) \\ c(x) & = & 1 + c(D(x)) \\ c(s_1 \,\mathsf{op}\, s_2) & = & 1 + max\{c(s_1), c(s_2)\},\ \mathsf{op} \in \{+, \|\} \end{array}$$

In the CSC approach a continuation is an application-specific structure of computations. The language MCC provides a general combination of parallel and sequential composition. As shown in [13], in order to model this combination of concepts a CSC continuation must be a tree of computations with active computations at the leaves. The major issue that gives rise to a tree-like structure is the presence of statements such as $(s_1 \parallel s_2); s_3$. In such a statement, s_3 can only execute after both s_1 and s_3 have terminated. Following [13], in order to define such domains of trees of computations we use a (partially ordered) set of *identifiers*.

DEFINITION 3.3

(a) Let $(\alpha \in)Id = \{1,2\}^*$ *be a set of* identifiers*, equipped with the following partial ordering:* $\alpha \le \alpha$*' iff* $\alpha' = \alpha \cdot i_1 \cdots i_n$ *for* $i_1, \cdots, i_n \in \{1,2\}, n \ge 0$.

(b) We define a function max : $\mathcal{P}(Id) \to \mathcal{P}(Id)$ *by:*

$$max(A) = \{\alpha \mid \alpha \,\text{is a maximal element of}\, (A, \le_A)\}$$

where $A \in \mathcal{P}(Id)$ *and* $\le_A$ *is the restriction of* $\le$ *to the subset* A *of* Id.

Id is the set of all finite, possibly empty, sequences over $\{1,2\}$ and $\alpha \le \alpha'$ iff α is a prefix of α'. We use the symbol λ to represent the empty sequence over $\{1,2\}$ (i.e. $\lambda \in Id = \{1,2\}^*$). Throughout this paper we use the symbol '$\cdot$' as a concatenation operator over sequences.

We can represent tree-like structures over $(Id, \le)$. Let $A = \{\alpha, \alpha\cdot 1, \alpha\cdot 2, \alpha\cdot 1\cdot 1, \alpha\cdot 1\cdot 2, \alpha\cdot 2\cdot 1, \alpha\cdot 2\cdot 2\}$; the maximal elements of $(A, \le_A)$ are exactly the *leaves* of the tree: $max(A) = \{\alpha\cdot 1\cdot 1, \alpha\cdot 1\cdot 2, \alpha\cdot 2\cdot 1, \alpha\cdot 2\cdot 2\}$. The elements of the set Id are used to distinguish between multiple occurrences of a computation in a continuation. The partial ordering $\le$ defined on Id is used to implement the semantics of sequential composition in the presence of parallelism as explained above.

Let $(\pi \in)\Pi = \mathcal{P}_{fin}(Id)$ and let $(x \in)\mathbf{X}$ be a metric domain, i.e. a complete metric space. We use the following notation: $\{\!|\mathbf{X}|\!\} \stackrel{not.}{=} \Pi \times (Id \to \mathbf{X})$. Let $\alpha \in Id, (\pi, \phi) \in \{\!|\mathbf{X}|\!\}$ with $\pi \in \Pi$, $\phi \in Id \to \mathbf{X}$. We define $id : \{\!|\mathbf{X}|\!\} \to \Pi$, $id(\pi, \phi) = \pi$. We also use the following abbreviations:

$$\begin{array}{rcll} (\pi,\phi)(\alpha) & \stackrel{not.}{=} & \phi(\alpha) & (\in \mathbf{X}) \\ (\pi,\phi) \setminus \pi' & \stackrel{not.}{=} & (\pi \setminus \pi', \phi) & (\in \{\!|\mathbf{X}|\!\}) \\ [\,(\pi,\phi) \mid \alpha \mapsto x\,] & \stackrel{not.}{=} & (\pi \cup \{\alpha\}, [\,\phi \mid \alpha \mapsto x\,]) & (\in \{\!|\mathbf{X}|\!\}) \end{array}$$

The basic idea is that we treat (π, ϕ) as a 'function' with finite graph $\{(\alpha, \phi(\alpha)) \mid \alpha \in \pi\}$, thus ignoring the behaviour of ϕ for any $\alpha \notin \pi$ (π is the 'domain' of (π, ϕ)). We find convenient to use this mathematical structure to represent finite partially ordered *bags* (or multisets)[2] of computations. As mentioned before, Id is used to distinguish between multiple

[2]We avoid using the notion of a *partially ordered multiset* which is a more refined structure – see [4], or chapter 16 of [3].

occurrences of a computation in such a bag. We endow both sets Id and Π with discrete metrics; every set with a discrete metric is a complete ultrametric space. According to Definition 2.1 and Remark 1, $(Id \to \mathbf{X}, d_{Id\to\mathbf{X}})$ becomes also a domain (i.e., a complete ultrametric space). It follows that $\Pi\times(Id \to \mathbf{X})$ becomes a domain. The metric over $\{\!|\mathbf{X}|\!\} = \Pi\times(Id \to \mathbf{X})$ is $d' = d_\Pi \times d_{Id\to\mathbf{X}}$ given by $d'((\pi,\phi),(\pi',\phi')) = max\{d_\Pi(\pi,\pi'), d_{Id\to\mathbf{X}}(\phi,\phi')\}$, where $d_{Id\to\mathbf{X}}(\phi,\phi') = sup_{\alpha\in Id} d_{\mathbf{X}}(\phi(\alpha),\phi'(\alpha))$ and the metric $d_{\mathbf{X}}$ over $\mathbf{X}$ is known. Intuitively, the operators behave as follows. $id(\pi,\phi)$ returns the collection of identifiers for the valid computations contained in the bag (π,ϕ), $(\pi,\phi)(\alpha)$ returns the computation with identifier α, $(\pi,\phi)\setminus\pi'$ removes the computations with identifiers in π', and $[\,(\pi,\phi) \mid \alpha \mapsto x\,]$ replaces the computation with identifier α.

By a slight abuse we use the same notations (including the operator id and the abbreviations $(\cdot)(\alpha), (\cdot)\setminus\pi, [\,\cdot \mid \alpha\mapsto x\,]$) when $(x\in)X$ is an ordinary set: $\{\!|X|\!\} = \Pi\times(Id\to X)$; in this case we do not endow $\{\!|X|\!\}$ with a metric structure.

4 Continuation Semantics for MCC

We design a continuation-based denotational semantics for MCC. As a semantic universe for the final yield of our denotational model we employ a standard linear-time domain $(p\in)\mathbf{P} = \mathcal{P}_{nco}(\mathbf{Q})$, where $\mathbf{Q} = \Sigma^*\cup\Sigma^*\cdot\{\delta\}\cup\Sigma^\infty$. Here $\Sigma^*(\Sigma^\infty)$ denotes the collection of all finite (infinite) sequences over Σ. $\mathbf{Q}$ contains finite sequences, possibly terminated with the symbol δ which denotes *deadlock*, or infinite sequences over Σ. We use the symbol λ to represent the empty sequence over Σ. This is a slight abuse, since we also use the symbol λ to represent the emptty sequence over $\{1,2\}$; however, it is always clear from the context which is the type of λ (either $\lambda\in Id = \{1,2\}^*$, or $\lambda\in\Sigma^*$). As it is well-known [3], by endowing Σ with the discrete metric we obtain a Baire metric on $\mathbf{Q}$. The type of the denotational semantics $\mathcal{D}$ for MCC is $Sem_D = Stat\to\mathbf{D}$, where:

$$
\begin{array}{rcl}
\mathbf{D} & \cong & \mathbf{Cont} \xrightarrow{1} \Sigma \to \mathbf{P} \\
(\gamma\in)\mathbf{Cont} & = & Id\times\mathbf{Kont} \\
(\kappa\in)\mathbf{Kont} & = & \{\!|\mathbf{Comp}|\!\} \\
\mathbf{Comp} & = & J^\diamondsuit + Snd + \frac{1}{2}\cdot\mathbf{D}
\end{array}
$$

Here $J^\diamondsuit = \{\diamondsuit\}\times J$, $Snd = Ch\times\Xi$ and $(\xi\in)\Xi = \Sigma\to\mathbb{Z}$. We use the notation $\langle j\rangle = (\diamondsuit, j)$. Also, for readability we denote typical elements (c,ξ) of Snd by $c!\xi$. In the equations above the sets Id (and Π), Σ, $J^\diamondsuit$ and Snd are endowed with the discrete metric, which is an ultrametric. The composed metric spaces are built up using the composite metrics of Definition 2.1. To conclude that such a system of equations has a (unique) solution (up to isometry) we rely on the general theory developed in [1]. The solution for $\mathbf{D}$ is obtained as a complete ultrametric space.

Comp is the domain of *computations*. A computation is either a (partially evaluated) denotation or a value used for (join) synchronization. The construction $\{\!|\mathbf{Comp}|\!\} = \Pi\times(Id\to\mathbf{Comp})$ was introduced in Section 3. In the sequel φ ranges over $Id\to\mathbf{Comp}$.

We call an element of **Kont** a *closed continuation* and an element of **Cont** an *open continuation*. A (closed or open) continuation is a representation of what remains to be computed from the program [12]. A closed continuation $\kappa \in \mathbf{Kont}$ is a self-contained structure of computations. An open continuation $(\alpha, \kappa) \in \mathbf{Cont}$ behaves like an evaluation context [6] for the denotational mapping. In an expression $\mathcal{D}(s)(\alpha, \kappa)$, $\mathcal{D}(s)$ is the *active computation* which is evaluated with respect to (α, κ). Intuitively, an open continuation (α, κ) is a structure of computations which contains a *hole*, indicating the conceptual position of the active computation. The position of the 'hole' is given in this representation by the identifier α, which is not an element of $id(\kappa)$ ($\mathcal{D}$ is designed to preserve this invariant property: $\alpha \notin id(\kappa)$ and $\alpha \in max(\{\alpha\} \cup id(\kappa))$).

The denotational function $\mathcal{D}$ is given in Definition 4.1 with the aid of a mapping kc, which is called a *scheduler*. The denotational function maps an open continuation to a program behavior. After producing an elementary step the denotational function transmits the control to the scheduler. The scheduler receives as parameter a closed continuation which it maps to a corresponding program behavior. If the continuation is empty the scheduler terminates the computation. If the continuation is not empty but there is no possibility to continue the computation then the scheduler detects a deadlock. Otherwise the scheduler can either activate a computation (denotation) or it can perform a join synchronization. In order to activate a computation the scheduler decomposes a closed continuation into a computation and a corresponding open continuation and then it executes the former with the latter as continuation. The activation of a computation can follow after zero or more (a finite number of) join synchronization steps.

We introduce three auxiliary mappings that are used by the scheduler. A *schedule* is a finite and non-empty set of identifiers. A schedule with only 1 element is an *activation schedule*; it contains exactly one identifier α, and the computation with identifier α is activated by the scheduler function kc. A schedule that contains more than 1 element is a *synchronization schedule*; it is used to define a pattern matching synchronization on multiple channels. Activation schedules are computed by the mapping Ω_A; synchronization schedules are computed by the mapping Ω_S. For all $\kappa \in \mathbf{Kont}$ we have $\Omega_A(\kappa) \cap \Omega_S(\kappa) = \emptyset$.

Let $(\varsigma \in) Sched = \mathcal{P}_{nfin}(Id)$ be the set of schedules. We use different symbols to represent typical elements of the sets $(\varsigma \in) Sched$ and $(\pi \in) \Pi$ so it is always clear when a (non-empty and) finite set of identifiers is treated as a schedule.[3] Let $\widehat{\cdot} : Sched \to \Pi$, $\widehat{\varsigma} = \varsigma$. With the aid of an auxiliary predicate $Sync : (Sched \times \mathbf{Kont}) \to Bool$ we define $\Omega, \Omega_A, \Omega_S : \mathbf{Kont} \to \mathcal{P}_{fin}(Sched)$ as follows:

$$Sync(\{\alpha\}, \kappa) = false$$

$$\begin{aligned} Sync(\{\alpha, \alpha_1, \ldots, \alpha_n\}, \kappa) = {} & (\kappa(\alpha) = \langle c_1?v_1 \,\&\, \cdots \,\&\, c_n?v_n\rangle \in J^{\diamondsuit}) \wedge \\ & (\kappa(\alpha_1) = c_1!\xi_1 \in Snd) \wedge \cdots \wedge (\kappa(\alpha_n) = c_n!\xi_n \in Snd) \end{aligned}$$

$$\Omega(\kappa) = \Omega_A(\kappa) \cup \Omega_S(\kappa)$$

[3] $\Pi = Sched \cup \{\emptyset\}$; the set $\Pi = \mathcal{P}_{fin}(Id)$ was introduced in Section 3.

$$\Omega_A(\kappa) = \{\{\alpha\} \mid \alpha \in max(id(\kappa)), \kappa(\alpha) \in \tfrac{1}{2} \cdot \mathbf{D}\}$$

$$\Omega_S(\kappa) = \{\varsigma \mid \varsigma \subseteq max(id(\kappa)), Sync(\varsigma, \kappa)\}$$

We use the mapping $(\cdot \mid \cdot \Rightarrow \cdot) : (\Sigma \times \Pi \times \mathbf{Kont}) \to \Sigma$ to model the state update effect of a multichannel synchronous communication:

$$(\,\sigma \mid \varsigma \Rightarrow \kappa\,) = \begin{cases} [\,\sigma \mid v_1 \mapsto \xi_1(\sigma) \mid \cdots \mid v_n \mapsto \xi_n(\sigma)\,] & \\ & \text{if } \varsigma \subseteq max(id(\kappa)), Sync(\varsigma, \kappa), \\ & \quad \varsigma = \{\alpha, \alpha_1, \ldots, \alpha_n\}, \\ & \quad \kappa(\alpha) = \langle c_1?v_1 \,\&\cdots\&\, c_n?v_n\rangle, \\ & \quad \kappa(\alpha_1) = c_1!\xi_1, \ldots, \kappa(\alpha_n) = c_n!\xi_n \\ \sigma & \text{otherwise} \end{cases}$$

We also define the predicates $Terminates, Blocks : \mathbf{Kont} \to Bool$ by:

$$Terminates(\kappa) = (id(\kappa) = \emptyset)$$

$$Blocks(\kappa) = (id(\kappa) \neq \emptyset) \wedge (\Omega(\kappa) = \emptyset)$$

DEFINITION 4.1 *(Denotational Semantics ($\mathcal{D}$) for MCC)*

(a) Let $kc : \mathbf{Kont} \to \Sigma \to \mathbf{P}$ be given by:

$$\begin{aligned} kc(\kappa)(\sigma) = {} & \text{if } Terminates(\kappa) \text{ then } \{\lambda\} \\ & \text{else if } Blocks(\kappa) \text{ then } \{\delta\} \\ & \text{else } \textstyle\bigcup_{\{\alpha\} \in \Omega_A(\kappa)} \quad \kappa(\alpha)(\alpha, \kappa \setminus \{\alpha\})(\sigma) \quad \cup \\ & \quad \textstyle\bigcup_{\varsigma \in \Omega_S(\kappa)} \quad (\,\sigma \mid \varsigma \Rightarrow \kappa\,) \cdot kc(\kappa \setminus \widehat{\varsigma})(\,\sigma \mid \varsigma \Rightarrow \kappa\,) \end{aligned}$$

(b) We define $\Phi : Sem_D \to Sem_D$ (for $(S \in) Sem_D = Stat \to \mathbf{D}$) by:

$$\begin{array}{rcl} \Phi(S)(\,\mathsf{skip}\,)(\alpha, \kappa)(\sigma) & = & \sigma \cdot kc(\kappa)(\sigma) \\ \Phi(S)(v := e)(\alpha, \kappa)(\sigma) & = & \sigma' \cdot kc(\kappa)(\sigma') \quad \text{where } \sigma' = [\,\sigma \mid v \mapsto \mathcal{E}[\![e]\!](\sigma)\,] \\ \Phi(S)(c!e)(\alpha, \kappa)(\sigma) & = & \sigma \cdot kc[\,\kappa \mid \alpha \mapsto c!\mathcal{E}[\![e]\!]\,](\sigma) \\ \Phi(S)(j)(\alpha, \kappa)(\sigma) & = & \sigma \cdot kc[\,\kappa \mid \alpha \mapsto \langle j\rangle\,](\sigma) \\ \Phi(S)(x)(\alpha, \kappa)(\sigma) & = & \Phi(S)(D(x))(\alpha, \kappa)(\sigma) \\ \Phi(S)(s_1 + s_2)(\alpha, \kappa)(\sigma) & = & \Phi(S)(s_1)(\alpha, \kappa)(\sigma) \;\cup\; \Phi(S)(s_2)(\alpha, \kappa)(\sigma) \\ \Phi(S)(s_1; s_2)(\alpha, \kappa)(\sigma) & = & \Phi(S)(s_1)(\alpha \cdot 1, [\,\kappa \mid \alpha \mapsto S(s_2)\,])(\sigma) \\ \Phi(S)(s_1 \parallel s_2)(\alpha, \kappa)(\sigma) & = & \Phi(S)(s_1)(\alpha \cdot 1, [\,\kappa \mid \alpha \cdot 2 \mapsto S(s_2)\,])(\sigma) \quad \cup \\ & & \Phi(S)(s_2)(\alpha \cdot 2, [\,\kappa \mid \alpha \cdot 1 \mapsto S(s_1)\,])(\sigma) \end{array}$$

(c) *We put* $\mathcal{D} = fix(\Phi)$. *Let* $\alpha_0 = \lambda$ *and* $\kappa_0 = (\emptyset, \varphi_0)$, *where* $\varphi_0(\alpha) = \mathcal{D}(\,\mathsf{skip}\,)$, $\forall\alpha \in Id$. *We also define* $\mathcal{D}[\![\cdot]\!] : Stat \to \Sigma \to \mathbf{P}$ *by:* $\mathcal{D}[\![s]\!] = \mathcal{D}(s)(\alpha_0, \kappa_0)$.

In the case of MCC a CSC continuation is a tree of computations with active elements at the leaves (the maximal elements with respect to '$\leq$'). In the case of a sequential composition $(s_1; s_2)$ the computations $\mathcal{D}(s_1)$ and $\mathcal{D}(s_2)$ are given the identifiers $\alpha \cdot 1$ and α, respectively $(\alpha \cdot 1 > \alpha)$. The scheduler function kc gives priority to the computations at the leaves of the tree that represents the continuation. Therefore $\mathcal{D}(s_2)$ will only be evaluated after the completion of the evaluation of $\mathcal{D}(s_1)$. In the case of a parallel composition $(s_1 \parallel s_2)$ the computations $\mathcal{D}(s_1)$ and $\mathcal{D}(s_2)$ are given the identifiers $\alpha \cdot 1$ and $\alpha \cdot 2$, respectively; $\alpha \cdot 1$ and $\alpha \cdot 2$ are incomparable (with respect to $\leq$) therefore the computations $\mathcal{D}(s_1)$ and $\mathcal{D}(s_2)$ are evaluated in an interleaved manner.

It may not be obvious that the scheduler function kc is well-defined, since it occurs in the right-hand side of its definition. The recursive occurrence of kc is preceded by the (contracting) synchronization step "$(\,\sigma \mid \varsigma \Rightarrow \kappa\,) \cdot \ldots$". We could define kc as fixed-point of an appropriate higher-order contraction. However, it is easier to define kc by induction on the number of communication patterns (of the type $J^{\diamondsuit}$) that are contained in the continuation κ. A continuation is a *finite* structure of **Comp** computations ($\mathbf{Comp} = J^{\diamondsuit} + Snd + \frac{1}{2} \cdot \mathbf{D}$). After each synchronization step the number of $J^{\diamondsuit}$ communication patterns that are contained in a continuation decreases by 1. When the continuation contains no communication pattern then $\Omega_S(\kappa) = \emptyset$. If $\Omega_S(\kappa) = \emptyset$ then the evaluation either terminates, or blocks, or $\Omega_A(\kappa) \neq \emptyset$. If $\Omega_A(\kappa) \neq \emptyset$ then a computation (of the type $\frac{1}{2} \cdot \mathbf{D}$) contained in κ is activated for evaluation.

LEMMA 4.1

(a) *The mapping kc (as introduced in 4.1) is well-defined.*

(b) $\forall \kappa_1, \kappa_2 \in \mathbf{Kont} : d(kc(\kappa_1), kc(\kappa_2)) \leq 2 \cdot d(\kappa_1, \kappa_2)$

LEMMA 4.2 *For all* $S \in Sem_D, s \in Stat, \alpha \in Id, \kappa \in \mathbf{Kont}, \sigma \in \Sigma$:

(a) $\Phi(S)(s)(\alpha, \kappa)(\sigma) \in \mathbf{P}$ *(it is well defined),*

(b) $\Phi(S)(s)$ *is nonexpansive (in* (α, κ)*), and*

(c) Φ *is* $\frac{1}{2}$*-contractive (in* S*).*

The proofs of 4.1 and 4.2 are omitted. Similar results are given in [14, 13].

EXAMPLE 4.1 *Let* $j \in J, s_1, s_2 \in Stat$, $j = c_1?v_1 \& c_2?v_2$, $s_1 = c_1!1 \parallel c_2!2$ *and* $s_2 = j; (v := v_1 + v_2 \parallel v := 10)$. *Let also* $\sigma_1 = [\,\sigma \mid v_1 \mapsto 1 \mid v_2 \mapsto 2\,]$, $\sigma_2 = [\,\sigma_1 \mid v \mapsto 3\,]$, $\sigma_3 = [\,\sigma_1 \mid v \mapsto 10\,]$ $(\sigma, \sigma_1, \sigma_2, \sigma_3 \in \Sigma)$.

One can check that $\mathcal{D}[\![s_1 \parallel s_2]\!](\sigma) = \{\sigma\sigma\sigma\sigma_1\sigma_2\sigma_3, \sigma\sigma\sigma\sigma_1\sigma_3\sigma_2\}$

5 Concurrency Laws

We present a technique of describing the behaviour of concurrent systems in a denotational model designed with CSC, using a representation of continuations as structures of computations (denotations). For MCC we prove that the semantic operators designed with CSC continuations satisfy the basic concurrency laws mentioned in the introduction. The semantics of nondeterministic choice '+' is given by the standard union operator, which is associative, commutativity and idempotent. Also, it is easy to prove the distributivity of the non-deterministic choice over sequential composition.

$$\begin{aligned}
&\mathcal{D}((s_1 + s_2); s_3)(\alpha, \kappa)(\sigma) \\
&= \mathcal{D}(s_1 + s_2)(\alpha \cdot 1, [\, \kappa \mid \alpha \mapsto \mathcal{D}(s_3)\,])(\sigma) \\
&= \mathcal{D}(s_1)(\alpha \cdot 1, [\, \kappa \mid \alpha \mapsto \mathcal{D}(s_3)\,])(\sigma) \;\cup\; \mathcal{D}(s_2)(\alpha \cdot 1, [\, \kappa \mid \alpha \mapsto \mathcal{D}(s_3)\,])(\sigma) \\
&= \mathcal{D}(s_1; s_3)(\alpha, \kappa)(\sigma) \;\cup\; \mathcal{D}(s_2; s3)(\alpha, \kappa)(\sigma) \\
&= \mathcal{D}((s_1; s_3) + (s_2; s_3))(\alpha, \kappa)(\sigma)
\end{aligned}$$

This implies $\mathcal{D}((s_1 + s_2); s_3) = \mathcal{D}(s_1; s_3 + s_2; s_3)$, because the proof is given for arbitrary $(\alpha, \kappa) \in \mathbf{Cont}$ and $\sigma \in \Sigma$.

But the flexibility provided by continuations comes at a price. The other properties of sequential and parallel composition require more complex arguments. We introduce and use the notion of a *resumption*. In this paper, a resumption is a structure of MCC statements.[4] A continuation can contain arbitrary values of the type $\mathbf{D}$. We prove the properties of sequential and parallel composition for continuations that can be obtained as semantified versions of resumptions, i.e. for continuations that contain *only* denotations of MCC statements. The function K defined in 5.1(b) maps a resumption to a corresponding continuation. We also introduce a notion of *isomorphism* over resumptions.

DEFINITION 5.1

(a) *Let* $(\theta \in)Comp = J^{\diamondsuit} \cup Snd \cup Stat$. *Notice that* $J \subseteq Stat$, *but* $J^{\diamondsuit} \cap Stat = \emptyset$ *(the set* $J^{\diamondsuit}$ *was defined at the beginning of section 4). We define the set of* closed resumptions $(k \in)KRes = \{\!|Comp|\!\}$.[5] *We define the set* $CRes$ *of* open resumptions *by:*

$$CRes = \{(\alpha, k) \mid (\alpha, k) \in (Id \times KRes), \alpha \notin id(k), \alpha \in max(\{\alpha\} \cup id(k))\}$$

[4] The terminology is similar to the one employed in [3], where the term resumption is used as an operational counterpart of the term continuation. For example, in [3] in the case of a sequential language a resumption may be a list (sequence) of program statements. However, in this technical report we do not present an operational semantics; we only present a denotational semantics.

[5] In this case the construct $\{\!| \cdot |\!\}$ is used to define an ordinary set; see the explanation given in the final part of section 3. The sets $J^{\diamondsuit}, Snd$ and $Stat$ are disjoint.

(b) *Let $\Theta : Comp \to \mathbf{Comp}$, $\Theta(\langle j\rangle) = \langle j\rangle, \Theta(c!\xi) = c!\xi$ and $\Theta(s) = \mathcal{D}(s)$. We define $K : KRes \to \mathbf{Kont}$ by $K(k) = (id(k), \varphi)$, where $\varphi(\alpha) = \Theta(k(\alpha)))$, $\forall \alpha \in Id$.*

We define the mappings $\omega, \omega_A, \omega_S : KRes \to \mathcal{P}_{fin}(Sched)$[6] with the aid of an auxiliary predicate $sync : (Sched \times KRes) \to Bool$

$$sync(\{\alpha\}, k) = false$$

$$sync(\{\alpha, \alpha_1, \ldots, \alpha_n\}, k) = (k(\alpha) = \langle c_1?v_1 \,\&\cdots\&\, c_n?v_n\rangle \in J^{\diamondsuit}) \wedge$$
$$(k(\alpha_1) = c_1!\xi_1 \in Snd) \wedge \cdots \wedge (k(\alpha_n) = c_n!\xi_n \in Snd)$$

$$\omega(k) = \omega_A(k) \cup \omega_S(k)$$

$$\omega_A(k) = \{\{\alpha\} \mid \alpha \in max(id(k)), k(\alpha) \in Stat\}$$

$$\omega_S(k) = \{\varsigma \mid \varsigma \subseteq max(id(k)), sync(\varsigma, k)\}$$

We define $(\,\cdot \mid \cdot \to \cdot\,) : (\Sigma \times Sched \times KRes) \to \Sigma$ as follows:

$$(\,\sigma \mid \varsigma \to k\,) = \begin{cases} [\,\sigma \mid v_1 \mapsto \xi_1(\sigma) \mid \cdots \mid v_n \mapsto \xi_n(\sigma)\,] & \\ & \text{if } \varsigma \subseteq max(id(k)), sync(\varsigma, k), \\ & \quad \varsigma = \{\alpha, \alpha_1, \ldots, \alpha_n\}, \\ & \quad k(\alpha) = \langle c_1?v_1 \,\&\cdots\&\, c_n?v_n\rangle, \\ & \quad k(\alpha_1) = c_1!\xi_1, \ldots, k(\alpha_n) = c_n!\xi_n \\ \sigma & \text{otherwise} \end{cases}$$

We aso define the predicates $terminates, blocks : KRes \to Bool$ by

$$terminates(k) = (id(k) = \emptyset)$$

$$blocks(k) = (id(k) \neq \emptyset) \wedge (\omega(k) = \emptyset)$$

Some properties of these mappings are stated in lemmas 5.6 and 5.7.

DEFINITION 5.2

(a) *We say that two closed resumptions $k_1, k_2 \in KRes$ are* isomorphic*, and we write $k_1 \cong k_2$, iff there is a bijection $\mu : id(k_1) \to id(k_2)$ such that:*

   (i) $\mu(\alpha_1) \leq \mu(\alpha_2) \Leftrightarrow \alpha_1 \leq \alpha_2, \quad \forall \alpha_1, \alpha_2 \in id(k_1)$

   (ii) $k_2(\mu(\alpha)) = k_1(\alpha), \quad \forall \alpha \in id(k_1)$

(b) *We say that two open resumptions $(\alpha_1, k_1), (\alpha_2, k_2) \in CRes$ are* isomorphic*, and we write $(\alpha_1, k_1) \cong (\alpha_2, k_2)$ iff there is a bijection $\mu : (\{\alpha_1\} \cup id(k_1)) \to (\{\alpha_2\} \cup id(k_2))$ such that:*

[6] $(\varsigma \in) Sched = \mathcal{P}_{nfin}(Id)$, see section 4.

(i) $\mu(\alpha_1) = \alpha_2$

(ii) $\mu(\alpha') \leq \mu(\alpha'') \Leftrightarrow \alpha' \leq \alpha'', \quad \forall \alpha', \alpha'' \in \{\alpha_1\} \cup id(k_1)$

(iii) $k_2(\mu(\alpha)) = k_1(\alpha), \quad \forall \alpha \in id(k_1)$

Obviously, $\forall (\alpha, k) \in CRes : (\alpha, k) \cong (\alpha, k)$ and if $(\alpha_1, k_1), (\alpha_2, k_2) \in CRes$ then $(\alpha_1, k_1) \cong (\alpha_2, k_2) \Rightarrow k_1 \cong k_2$. Also, the following Lemma is easily established.

LEMMA 5.1 *For all $k \in KRes, \alpha \in Id, \theta \in Comp$:*

$$K[\, k \mid \alpha \mapsto \theta \,] = [\, K(k) \mid \alpha \mapsto \Theta(\theta) \,]$$

In Corollary 1 we show that any two continuations that correspond to isomorphic resumptions behave the same. This result is obtained by combining Lemma 5.2 with an argument '$\epsilon \leq \frac{1}{2} \cdot \epsilon \Rightarrow \epsilon = 0$'. Lemma 5.2 identifies the property (in this case the isomorphism between resumptions) which is preserved at each computation step. The effect of each computation step is given in the present metric setting by the $\frac{1}{2} \cdot$ -contracting factor that appears in the statement of Lemma 5.2(b).

In the proof of Lemma 5.2 we use some results presented later in Lemma 5.6 and Lemma 5.7.

LEMMA 5.2

(a) For all $k_1, k_2 \in KRes$ with $k_1 \cong k_2$ and $\sigma \in \Sigma$, there exists $\overline{s} \in Stat$, $(\overline{\alpha}_1, \overline{k}_1), (\overline{\alpha}_2, \overline{k}_2) \in CRes$ with $(\overline{\alpha}_1, \overline{k}_1) \cong (\overline{\alpha}_2, \overline{k}_2)$ and $\overline{\sigma} \in \Sigma$ such that:

$$\begin{aligned} & d(kc(K(k_1)(\sigma)), kc(K(k_2))(\sigma)) \leq \\ & \qquad d(\mathcal{D}(\overline{s})(\overline{\alpha}_1, K(\overline{k}_1))(\overline{\sigma}), \mathcal{D}(\overline{s})(\overline{\alpha}_2, K(\overline{k}_2))(\overline{\sigma})) \end{aligned}$$

(b) For all $s \in Stat, (\alpha_1, k_1), (\alpha_2, k_2) \in CRes$ with $(\alpha_1, k_1) \cong (\alpha_2, k_2)$ and $\sigma \in \Sigma$, there exists $\overline{s} \in Stat, (\overline{\alpha}_1, \overline{k}_1), (\overline{\alpha}_2, \overline{k}_2) \in CRes$ with $(\overline{\alpha}_1, \overline{k}_1) \cong (\overline{\alpha}_2, \overline{k}_2)$ and $\overline{\sigma} \in \Sigma$ such that:

$$\begin{aligned} & d(\mathcal{D}(s)(\alpha_1, K(k_1))(\sigma), \mathcal{D}(s)(\alpha_2, K(k_2))(\sigma)) \leq \\ & \qquad \tfrac{1}{2} \cdot d(\mathcal{D}(\overline{s})(\overline{\alpha}_1, K(\overline{k}_1))(\overline{\sigma}), \mathcal{D}(\overline{s})(\overline{\alpha}_2, K(\overline{k}_2))(\overline{\sigma})) \end{aligned}$$

Proof: For Lemma 5.2(a) we distinguish the following subcases:

- Case $terminates(k_1)$. By Lemma 5.7(d) we also have $terminates(k_2)$. In this case by 5.6(e):

$$d(kc(K(k_1))(\sigma), kc(K(k_2))(\sigma)) = d(\{\lambda\}, \{\lambda\}) = 0$$

- Case $\neg terminates(k_1) \wedge blocks(k_1)$. By 5.7(d) and 5.7(e) we also have $\neg terminates(k_2) \wedge blocks(k_2)$. By 5.6(e) and 5.6(f):

$$d(kc(K(k_1))(\sigma), kc(K(k_2))(\sigma)) = d(\{\delta\}, \{\delta\}) = 0$$

- Case $\neg terminates(k_1) \wedge \neg blocks(k_1)$. In this case we also have $\neg terminates(k_2)$ and $\neg blocks(k_2)$ and $\omega(k_1) \neq \emptyset$ and $\omega(k_2) \neq \emptyset$.

$$\begin{aligned}
& d(kc(K(k_1))(\sigma), kc(K(k_2))(\sigma)) \\
& = d(\textstyle\bigcup_{\{\alpha\}\in\Omega_A(K(k_1))} K(k_1)(\alpha)(\alpha, K(k_1) \setminus \{\alpha\})(\sigma) \cup \\
& \quad \textstyle\bigcup_{\varsigma\in\Omega_S(K(k_1))} (\, \sigma \mid \varsigma \Rightarrow K(k_1)\,) \cdot kc(K(k_1) \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow K(k_1)\,), \\
& \quad \textstyle\bigcup_{\{\alpha\}\in\Omega_A(K(k_2))} K(k_2)(\alpha)(\alpha, K(k_2) \setminus \{\alpha\})(\sigma) \cup \\
& \quad \textstyle\bigcup_{\varsigma\in\Omega_S(K(k_2))} (\, \sigma \mid \varsigma \Rightarrow K(k_2)\,) \cdot kc(K(k_2) \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow K(k_2)\,)) \\
& \text{[Lemma 5.6(b), 5.6(c)]} \\
& = d(\textstyle\bigcup_{\{\alpha\}\in\omega_A(k_1)} K(k_1)(\alpha)(\alpha, K(k_1) \setminus \{\alpha\})(\sigma) \cup \\
& \quad \textstyle\bigcup_{\varsigma\in\omega_S(k_1)} (\, \sigma \mid \varsigma \Rightarrow K(k_1)\,) \cdot kc(K(k_1) \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow K(k_1)\,), \\
& \quad \textstyle\bigcup_{\{\alpha\}\in\omega_A(k_2)} K(k_2)(\alpha)(\alpha, K(k_2) \setminus \{\alpha\})(\sigma) \cup \\
& \quad \textstyle\bigcup_{\varsigma\in\omega_S(k_2)} (\, \sigma \mid \varsigma \Rightarrow K(k_2)\,) \cdot kc(K(k_2) \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow K(k_2)\,))) \\
& \text{[Lemma 5.7(a), 5.7(b), Lemma 5.6(j)]} \\
& = d(\textstyle\bigcup_{\{\alpha\}\in\omega_A(k_1)} K(k_1)(\alpha)(\alpha, K(k_1) \setminus \{\alpha\})(\sigma) \cup \\
& \quad \textstyle\bigcup_{\varsigma\in\omega_S(k_1)} (\, \sigma \mid \varsigma \rightarrow k_1\,) \cdot kc(K(k_1) \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \rightarrow k_1\,), \\
& \quad \textstyle\bigcup_{\{\alpha\}\in\omega_A(k_1)} K(k_2)(\mu(\alpha))(\mu(\alpha), K(k_2) \setminus \{\mu(\alpha)\})(\sigma) \cup \\
& \quad \textstyle\bigcup_{\varsigma\in\omega_S(k_1)} (\, \sigma \mid \mu(\varsigma) \rightarrow k_2\,) \cdot kc(K(k_2) \setminus \widehat{\mu(\varsigma)}\,)(\, \sigma \mid \mu(\varsigma) \rightarrow k_2\,))) \\
& \text{['$\cup$' is nonexpansive, Lemma 5.7(j)]} \\
& \leq max\{ \\
& \quad max\{d(K(k_1)(\alpha)(\alpha, K(k_1) \setminus \{\alpha\})(\sigma), K(k_2)(\mu(\alpha))(\mu(\alpha), K(k_2) \setminus \{\mu(\alpha)\})(\sigma)) \\
& \qquad\quad \mid \{\alpha\} \in \omega_A(k_1)\}, \\
& \quad max\{d((\, \sigma \mid \varsigma \rightarrow k_1\,) \cdot kc(K(k_1) \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \rightarrow k_1\,), \\
& \qquad\qquad (\, \sigma \mid \varsigma \rightarrow k_1\,) \cdot kc(K(k_2) \setminus \widehat{\mu(\varsigma)}\,)(\, \sigma \mid \varsigma \rightarrow k_1\,)) \\
& \qquad\quad \mid \varsigma \in \omega_S(k_1)\}\}^{\,(5.2.a.1)}
\end{aligned}$$

where $\mu : id(k_1) \rightarrow id(k_2)$ is a bijection that satisfies the properties given in definition 5.2(a), and we use the notation

$$\mu(\{\alpha_1, \ldots, \alpha_n\}) \overset{not.}{=} \{\mu(\alpha_1), \ldots, \mu(\alpha_n)\}$$

for any schedule $\{\alpha_1, \ldots, \alpha_n\} \in Sched$. We proceed by induction on $|I(k_1)|$, where

$$I(k_1) = \{\alpha \mid \alpha \in id(k_1), k_1(\alpha) \in J^{\diamondsuit}\}$$

If $|I(k_1)| = 0$ then, by Definition 5.1 and Lemma 5.6, $\omega_S(k_1) = \emptyset$. In this case, by Lemma 5.6(g) we have

$$\stackrel{(5.2.a.1)}{=} max\{d(\mathcal{D}(s)(\alpha, K(k_1 \setminus \{\alpha\}))(\sigma), \mathcal{D}(s)(\mu(\alpha), K(k_2 \setminus \{\mu(\alpha)\}))(\sigma)) \\ \mid \{\alpha\} \in \omega_A(k_1), s = k_1(\alpha) = k_2(\mu(\alpha))\}$$

As $k_1 \cong k_2, \forall \{\alpha\} \in \omega_A(k_1) : (\alpha, k_1 \setminus \{\alpha\}), (\mu(\alpha), k_2 \setminus \{\mu(\alpha)\}) \in CRes$ and $(\alpha, k_1 \setminus \{\alpha\}) \cong (\mu(\alpha), k_2 \setminus \{\mu(\alpha)\})$. In this subcase we conclude that $\exists \overline{s} \in Stat, \overline{\alpha} \in Id$ such that $\overline{s} = k_1(\overline{\alpha}) = k_2(\mu(\overline{\alpha})), (\overline{\alpha}, k_1 \setminus \{\overline{\alpha}\}) \cong (\mu(\overline{\alpha}), k_2 \setminus \{\mu(\overline{\alpha})\})$ and:

$$d(kc(K(k_1))(\sigma), kc(K(k_2))(\sigma)) \\ \leq d(\mathcal{D}(\overline{s})(\overline{\alpha}, K(k_1 \setminus \{\overline{\alpha}\}))(\sigma), \mathcal{D}(\overline{s})(\mu(\overline{\alpha}), K(k_2 \setminus \{\mu(\overline{\alpha})\}))(\sigma))$$

Next we assume that $|I(k_1)| > 0$. We have $\omega(k_1) = \omega_A(k_1) \cup \omega_S(k_1)$. By Lemma 5.6(b) and 5.6(c) $\omega_A(k_1) = \Omega_A(K(k_1))$, $\omega_S(k_1) = \Omega_S(K(k_1))$. In this case we have:

$$\stackrel{(5.2.a.1)}{=}$$

$$= max\{ \\ max\{d(K(k_1)(\alpha)(\alpha, K(k_1) \setminus \{\alpha\})(\sigma), K(k_2)(\mu(\alpha))(\mu(\alpha), K(k_2) \setminus \{\mu(\alpha)\})(\sigma)) \\ \mid \{\alpha\} \in \omega_A(k_1)\}^{(5.2.a.2)}, \\ max\{d((\, \sigma \mid \varsigma \to k_1\,) \cdot kc(K(k_1) \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \to k_1\,), \\ (\, \sigma \mid \varsigma \to k_1\,) \cdot kc(K(k_2) \setminus \widehat{\mu(\varsigma)}\,)(\, \sigma \mid \varsigma \to k_1\,)) \\ \mid \varsigma \in \omega_S(k_1)\}^{(5.2.a.3)} \}$$

If $\stackrel{(5.2.a.2)}{\geq} \stackrel{(5.2.a.3)}{}$ then we proceed as in the basic case (i.e. $|I(k_1)| = 0$) and we obtain the desired result. Now assume that $\stackrel{(5.2.a.2)}{<} \stackrel{(5.2.a.3)}{}$. By using Lemma 5.6(g) we can compute as follows:

$$\stackrel{(5.2.a.3)}{=}$$

$$= \frac{1}{2} \cdot max\{d(kc(K(k_1 \setminus \widehat{\varsigma}))(\, \sigma \mid \varsigma \to k_1\,), kc(K(k_2 \setminus \widehat{\mu(\varsigma)}\,))(\, \sigma \mid \varsigma \to k_1\,)) \\ \mid \varsigma \in \omega_S(k_1)\}$$

Notice that $\forall \varsigma \in Sched$

$$k_1 \setminus \widehat{\varsigma} \cong k_2 \setminus \widehat{\mu(\varsigma)}$$

and $|I(k_1 \setminus \widehat{\varsigma})| < |I(k_1)|$, hence, by the induction hypothesis $\exists \overline{s}' \in Stat, \overline{\sigma}' \in \Sigma, (\overline{\alpha}'_1, \overline{k}'_1), (\overline{\alpha}'_2, \overline{k}'_2) \in CRes$ such that $(\overline{\alpha}'_1, \overline{k}'_1) \cong (\overline{\alpha}'_2, \overline{k}'_2)$ and

$$d(kc(K(k_1 \setminus \widehat{\varsigma}))((\, \sigma \mid \varsigma \to k_1\,)), kc(K(k_2 \setminus \widehat{\mu(\varsigma)}\,))((\, \sigma \mid \varsigma \to k_1\,))) \\ \leq d(\mathcal{D}(\overline{s}')(\overline{\alpha}'_1, K(\overline{k}'_1))(\overline{\sigma}'), \mathcal{D}(\overline{s}')(\overline{\alpha}'_2, K(\overline{k}'_2))(\overline{\sigma}'))$$

It follows immediately that $\exists \overline{s} \in Stat, \overline{\sigma} \in \Sigma, (\overline{\alpha}_1, \overline{k}_1), (\overline{\alpha}_2, \overline{k}_2) \in CRes$ such that $(\overline{\alpha}_1, \overline{k}_1) \cong (\overline{\alpha}_2, \overline{k}_2)$ and

$$\stackrel{(5.2.a.3)}{\leq} d(\mathcal{D}(\overline{s})(\overline{\alpha}_1, K(\overline{k}_1))(\overline{\sigma}), \mathcal{D}(\overline{s})(\overline{\alpha}_2, K(\overline{k}_2))(\overline{\sigma}))$$

Next we prove Lemma 5.2(b) by induction on $c(s)$. We treat three subcases.

Case $[s{=}j]$.

$d(\mathcal{D}(j)(\alpha_1, K(k_1))(\sigma), \mathcal{D}(j)(\alpha_2, K(k_2))(\sigma))$
$= d(\sigma \cdot kc[\, K(k_1) \mid \alpha_1 \mapsto \langle j\rangle\,](\sigma), \sigma \cdot kc[\, K(k_2) \mid \alpha_2 \mapsto \langle j\rangle\,](\sigma))$ [Lemma 5.1]
$= \frac{1}{2} \cdot\ d(kc(K[\, k_1 \mid \alpha_1 \mapsto \langle j\rangle\,])(\sigma), kc(K[\, k_2 \mid \alpha_2 \mapsto \langle j\rangle\,])(\sigma))\ ^{(5.2.b.1)}$

It is easy to check that $(\alpha_1, k_1) \cong (\alpha_2, k_2)$ implies $[\, k_1 \mid \alpha_1 \mapsto \langle j\rangle\,] \cong\ [\, k_2 \mid \alpha_2 \mapsto \langle j\rangle\,]$. Thus, by Lemma 5.2(a) $\exists \overline{s} \in \mathit{Stat}, \overline{\sigma} \in \Sigma, (\overline{\alpha}_1, \overline{k}_1), (\overline{\alpha}_2, \overline{k}_2) \in \mathit{CRes}$ such that $(\overline{\alpha}_1, \overline{k}_1) \cong (\overline{\alpha}_2, \overline{k}_2)$ and:

$^{(5.2.b.1)}\ \leq \frac{1}{2} \cdot\ d(\mathcal{D}(\overline{s})(\overline{\alpha}_1, K(\overline{k}_1))(\overline{\sigma}), \mathcal{D}(\overline{s})(\overline{\alpha}_2, K(\overline{k}_2))(\overline{\sigma}))$

Case $[s{=}x]$.

$d(\mathcal{D}(x)(\alpha_1, K(k_1))(\sigma), \mathcal{D}(x)(\alpha_2, K(k_2))(\sigma))$
$= d(\mathcal{D}(D(x))(\alpha_1, K(k_1))(\sigma), \mathcal{D}(D(x))(\alpha_2, K(k_2))(\sigma))\ ^{(5.2.b.2)}$

By the induction hypothesis $(c(x) < c(D(x)))$ $\exists \overline{s} \in \mathit{Stat}, \overline{\sigma} \in \Sigma, (\overline{\alpha}_1, \overline{k}_1), (\overline{\alpha}_2, \overline{k}_2) \in \mathit{CRes}$ with $(\overline{\alpha}_1, \overline{k}_1) \cong (\overline{\alpha}_2, \overline{k}_2)$ such that:

$^{(5.2.b.2)}\ \leq \frac{1}{2} \cdot\ d(\mathcal{D}(\overline{s})(\overline{\alpha}_1, K(\overline{k}_1))(\overline{\sigma}), \mathcal{D}(\overline{s})(\overline{\alpha}_2, K(\overline{k}_2))(\overline{\sigma}))$

Case $[s{=}s_1 \parallel s_2]$.

$$
\begin{aligned}
& d(\mathcal{D}(s_1 \parallel s_2)(\alpha_1, K(k_1))(\sigma), \mathcal{D}(s_1 \parallel s_2)(\alpha_2, K(k_2))(\sigma)) \\
& = d(\mathcal{D}(s_1)(\alpha_1 \cdot 1, [\, K(k_1) \mid \alpha_1 \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma) \cup \\
& \quad \mathcal{D}(s_2)(\alpha_1 \cdot 2, [\, K(k_1) \mid \alpha_1 \cdot 1 \mapsto \mathcal{D}(s_1)\,])(\sigma), \\
& \quad \mathcal{D}(s_1)(\alpha_2 \cdot 1, [\, K(k_2) \mid \alpha_2 \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma) \cup \\
& \quad \mathcal{D}(s_2)(\alpha_2 \cdot 2, [\, K(k_2) \mid \alpha_2 \cdot 1 \mapsto \mathcal{D}(s_1)\,])(\sigma))
\end{aligned}
$$

[Lemma 5.1, '$\cup$' is nonexpansive]

$$
\begin{aligned}
& \leq max\{d(\mathcal{D}(s_1)(\alpha_1 \cdot 1, K[\, k_1 \mid \alpha_1 \cdot 2 \mapsto s_2\,])(\sigma) \cup \\
& \qquad \mathcal{D}(s_1)(\alpha_2 \cdot 1, K[\, k_2 \mid \alpha_2 \cdot 2 \mapsto s_2\,])(\sigma))\ ^{(5.2.b.3)}, \\
& \qquad d(\mathcal{D}(s_2)(\alpha_1 \cdot 2, K[\, k_1 \mid \alpha_1 \cdot 1 \mapsto s_1\,])(\sigma) \cup \\
& \qquad \mathcal{D}(s_2)(\alpha_2 \cdot 2, K[\, k_2 \mid \alpha_2 \cdot 1 \mapsto s_1\,])(\sigma))\ ^{(5.2.b.4)}\ \}
\end{aligned}
$$

It is not difficult to check that $(\alpha_1, k_1)\ \cong\ (\alpha_2, k_2)$ implies

$(\alpha_1 \cdot 1, [\, k_1 \mid \alpha_1 \cdot 2 \mapsto s_2\,]) \cong (\alpha_2 \cdot 1, [\, k_2 \mid \alpha_2 \cdot 2 \mapsto s_2\,])$ and
$(\alpha_1 \cdot 2, [\, k_1 \mid \alpha_1 \cdot 1 \mapsto s_1\,]) \cong (\alpha_2 \cdot 2, [\, k_2 \mid \alpha_2 \cdot 1 \mapsto s_1\,])$.

For example, in the first case we can use a bijection that maps $\alpha_1 \cdot 1$ to $\alpha_2 \cdot 1$ and $\alpha_1 \cdot 2$ to $\alpha_2 \cdot 2$. Therefore, as $c(s_1) < c(s_1 \parallel s_2)$ we can use the induction hypothesis for $^{(5.2.b.3)}$ and we infer that $\exists \overline{s} \in \mathit{Stat}, \overline{\sigma} \in \Sigma, (\overline{\alpha}_1, \overline{k}_1), (\overline{\alpha}_2, \overline{k}_2) \in \mathit{CRes}$ such that $(\overline{\alpha}_1, \overline{k}_1) \cong (\overline{\alpha}_2, \overline{k}_2)$ and:

$\overset{(5.2.b.3)}{\leq} \frac{1}{2} \cdot d(\mathcal{D}(\overline{s})(\overline{\alpha}_1, K(\overline{k}_1))(\overline{\sigma}), \mathcal{D}(\overline{s})(\overline{\alpha}_2, K(\overline{k}_2))(\overline{\sigma}))$

The expression $^{(5.2.b.4)}$ can be handled in a similar manner and the desired result follows immediately. □

Corollary 1

(a) For all $s \in Stat, (\alpha_1, k_1), (\alpha_2, k_2) \in CRes$ *with* $(\alpha_1, k_1) \cong (\alpha_2, k_2), \sigma \in \Sigma$*:*

$$\mathcal{D}(s)(\alpha_1, K(k_1))(\sigma) = \mathcal{D}(s)(\alpha_2, K(k_2))(\sigma)$$

(b) For all $k_1, k_2 \in KRes$ *with* $k_1 \cong k_2$ *and* $\sigma \in \Sigma$*:*

$$kc(K(k_1))(\sigma) = kc(K(k_2))(\sigma)$$

Proof: Let $(w \in)W_{\mathcal{D}} = Stat \times CRes \times CRes \times \Sigma$. For $(s, (\alpha_1, k_1), (\alpha_2, k_2), \sigma) \in W_{\mathcal{D}}$ we use the notation:

$$\epsilon_{\mathcal{D}}(s, (\alpha_1, k_1), (\alpha_2, k_2), \sigma) \overset{not.}{=} d(\mathcal{D}(s)(\alpha_1, K(k_1))(\sigma), \mathcal{D}(s)(\alpha_2, K(k_2))(\sigma))$$

By lemma 5.2(b) for all $(s, (\alpha_1, k_1), (\alpha_2, k_2), \sigma) \in W_{\mathcal{D}}$ with $(\alpha_1, k_1) \cong (\alpha_2, k_2)$ there exists $(\overline{s}, (\overline{\alpha}_1, \overline{k}_1), (\overline{\alpha}_2, \overline{k}_2), \overline{\sigma}) \in W_{\mathcal{D}}$ with $(\overline{\alpha}_1, \overline{k}_1) \cong (\overline{\alpha}_2, \overline{k}_2)$ such that:

$$\epsilon_{\mathcal{D}}(s, (\alpha_1, k_1), (\alpha_2, k_2), \sigma) \leq \frac{1}{2} \cdot \epsilon_{\mathcal{D}}(\overline{s}, (\overline{\alpha}_1, \overline{k}_1), (\overline{\alpha}_2, \overline{k}_2), \overline{\sigma})$$

This means that we have $sup_{w \in W_{\mathcal{D}}} \epsilon_{\mathcal{D}}(w) \leq \frac{1}{2} \cdot sup_{\overline{w} \in W_{\mathcal{D}}} \epsilon_{\mathcal{D}}(\overline{w})$, where $w = (s, (\alpha_1, k_1), (\alpha_2, k_2), \sigma)$ and $\overline{w} = (\overline{s}, (\overline{\alpha}_1, \overline{k}_1), (\overline{\alpha}_2, \overline{k}_2), \overline{\sigma})$; this implies that $sup_{w \in W_{\mathcal{D}}} \epsilon_{\mathcal{D}}(w) = 0$, i.e., $d(\mathcal{D}(s)(\alpha_1, K(k_1))(\sigma), \mathcal{D}(s)(\alpha_2, K(k_2))(\sigma)) = 0$ and thus $\mathcal{D}(s)(\alpha_1, K(k_1))(\sigma) = \mathcal{D}(s)(\alpha_2, K(k_2))(\sigma)$, which proves 1(a). 1(b) follows immediately from Lemma 5.2(a) and Corollary 1(a). □

Corollary 2 *If* $\theta \in Comp, (\alpha, k) \in CRes, \alpha' \in Id, \alpha' \notin id(k)$ *(in general* $\alpha' \neq \alpha$*) then:*

(a) $\mathcal{D}(s)(\alpha, K(k))(\sigma) = \mathcal{D}(s)(\alpha, [\, K(k) \mid \alpha' \mapsto \Theta(\theta) \,] \setminus \{\alpha'\})(\sigma)$

(b) $kc(K(k))(\sigma) = kc([\, K(k) \mid \alpha' \mapsto \Theta(\theta) \,] \setminus \{\alpha'\})(\sigma)$

Proof: This follows easily by using Corollary 1(b) and Lemma 5.1 if we notice that

$$(\alpha, k) \cong (\alpha, [\, k \mid \alpha' \mapsto \theta \,] \setminus \{\alpha'\})$$

$$k \cong [\, k \mid \alpha' \mapsto \theta \,] \setminus \{\alpha'\}$$

□

Lemmas 5.3 and 5.4 can be approached with the same technique that was used in the proofs of Lemma 5.2 and Corollary 1, by establishing appropriate properties that are preserved by the computation steps and by using the '$\epsilon \leq \frac{1}{2} \cdot \epsilon \Rightarrow \epsilon = 0$' argument. In the proofs of 5.3 and 5.4 we use some results presented later in Lemma 5.6 and Lemma 5.7.

LEMMA 5.3 *For all* $s_0, s_1, s_2 \in Stat, \sigma \in \Sigma, \alpha_0, \alpha \in Id, k \in KRes$ *such that* $(\alpha_0, k) \in CRes, (\alpha, k) \in CRes$*,* $(\alpha_0 \neq \alpha), \neg(\alpha_0 \leq \alpha)$ *and* $\neg(\alpha \leq \alpha_0)$*, we have:*

(a) $kc([\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma) =$

$kc([\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma)$

(b) $\mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma) =$

$\mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma)$

Proof: We use the notation $Q(\alpha_0, \alpha)$, for $\alpha_0, \alpha \in Id$, to express the fact that α_0 and α are different and incomparable with respect to '$\leq$'.

$$Q(\alpha_0, \alpha) \stackrel{not.}{=} (\alpha_0 \neq \alpha) \wedge (\neg(\alpha_0 \leq \alpha)) \wedge (\neg(\alpha \leq \alpha_0))$$

Also, we use the notation

$$P_{\parallel}(\alpha_0, \alpha, k) \stackrel{not.}{=} Q(\alpha_0, \alpha) \wedge ((\alpha_0, k) \in CRes) \wedge ((\alpha, k) \in CRes)$$

$P_{\parallel}$ is the invariant property which is preserved by the computation steps.

We proceed as follows. For 5.3(a) we show that $\forall s_1, s_2 \in Stat, \sigma \in \Sigma, \alpha \in Id, k \in KRes$ with $(\alpha, k) \in CRes$, $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ with $P_{\parallel}(\alpha', \alpha, k')$ such that:

$$\begin{aligned}
&\epsilon^{\parallel}_{kc}(k, \alpha, s_1, s_2, \sigma) \stackrel{not.}{=} \\
&\quad d(kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,](\sigma), \\
&\qquad kc[\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,](\sigma)) \\
&\leq d(\mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma'), \\
&\qquad \mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma')) \quad (5.3.1) \\
&\stackrel{not.}{=} \epsilon^{\parallel}_{\mathcal{D}}(s', \alpha', k', \alpha, s_1, s_2, \sigma')
\end{aligned}$$

Also, for 5.3(b) we show that $\forall s_0, s_1, s_2 \in Stat, \sigma \in \Sigma, \alpha_0, \alpha \in Id, k \in KRes$ such that $P_{\parallel}(\alpha_0, \alpha, k)$, $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ with $P_{\parallel}(\alpha', \alpha, k')$ such that:

$$\begin{aligned}
&\epsilon^{\parallel}_{\mathcal{D}}(s_0, \alpha_0, k, \alpha, s_1, s_2, \sigma) \\
&= d(\mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma), \\
&\qquad \mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma)) \\
&\leq \tfrac{1}{2} \cdot\ d(\mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma'), \\
&\qquad\quad \mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma')) \quad (5.3.2) \\
&\stackrel{not.}{=} \epsilon^{\parallel}_{\mathcal{D}}(s', \alpha', k', \alpha, s_1, s_2, \sigma')
\end{aligned}$$

If we put $(w \in)W = Stat \times Id \times CRes \times Id \times Stat \times Stat \times \Sigma$ we infer that:

$$sup_{\ w\in W\,:\,P_{\|}(\alpha_0,\alpha,k)}\ \epsilon_{\mathcal{D}}^{\|}(w) \leq \tfrac{1}{2}\cdot\ sup_{\ w'\in W\,:\,P_{\|}(\alpha',\alpha,k')}\ \epsilon_{\mathcal{D}}^{\|}(w')$$

where $\quad w = (s_0, \alpha_0, k, \alpha, s_1, s_2, \sigma) \quad$ and $\quad w' = (s', \alpha', k', \alpha, s_1, s_2, \sigma')$.
However $sup_{\ w\in W\,:\,P_{\|}(\alpha_0,\alpha,k)}\ \epsilon_{\mathcal{D}}^{\|}(w) = sup_{\ w'\in W\,:\,P_{\|}(\alpha',\alpha,k')}\ \epsilon_{\mathcal{D}}^{\|}(w')$. So we infer that:

$$sup_{\ w\in W\,:\,P_{\|}(\alpha_0,\alpha,k)}\ \epsilon_{\mathcal{D}}^{\|}(w) = 0$$

which implies immediately 5.3(b). Next, by using this result and $^{(5.3.1)}$ we obtain immediately 5.3(a). In the sequel we prove $^{(5.3.1)}$ and $^{(5.3.2)}$.

First, we prove $^{(5.3.1)}$. Let $s_1, s_2 \in Stat, \sigma \in \Sigma, \alpha \in Id, k \in KRes$ with $(\alpha, k) \in CRes$. Let also $\kappa = [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,]$, $\overline{\kappa} = [\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,]$, $k'' = [\,k \mid \alpha \mapsto s_1 \parallel s_2\,]$ and $\overline{k}'' = [\,k \mid \alpha\cdot 1 \mapsto s_1 \mid \alpha\cdot 2 \mapsto s_2\,]$. By Lemma 5.1 $\kappa = K(k''), \overline{\kappa} = K(\overline{k}'')$. We compute as follows:

$$
\begin{aligned}
& d(kc(\kappa)(\sigma), kc(\overline{\kappa})(\sigma)) \\
&= d(kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,](\sigma), \\
&\qquad kc[\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_2)\,](\sigma)) \\
&= d(\mathcal{D}(s_1 \parallel s_2)(\alpha, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,] \setminus \{\alpha\})(\sigma) \cup \\
&\qquad \textstyle\bigcup_{\{\alpha'\}\in\Omega_A(K(k'')),\alpha'\neq\alpha} K(k'')(\alpha')(\alpha', K(k'') \setminus \{\alpha'\})(\sigma) \cup \\
&\qquad \textstyle\bigcup_{\varsigma\in\Omega_S(K(k''))} (\,\sigma \mid \varsigma \Rightarrow K(k'')\,) \cdot kc(K(k'') \setminus \widehat{\varsigma})(\,\sigma \mid \varsigma \Rightarrow K(k'')\,), \\
&\qquad \mathcal{D}(s_1)(\alpha\cdot 1, [\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_2)\,] \setminus \{\alpha\cdot 1\})(\sigma) \cup \\
&\qquad \mathcal{D}(s_2)(\alpha\cdot 2, [\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_2)\,] \setminus \{\alpha\cdot 2\})(\sigma) \cup \\
&\qquad \textstyle\bigcup_{\{\alpha'\}\in\Omega_A(K(\overline{k}'')),\alpha'\neq\alpha\cdot 1,\alpha'\neq\alpha\cdot 2} K(\overline{k}'')(\alpha')(\alpha', K(\overline{k}'') \setminus \{\alpha'\})(\sigma) \cup \\
&\qquad \textstyle\bigcup_{\varsigma\in\Omega_S(K(\overline{k}''))} (\,\sigma \mid \varsigma \Rightarrow K(\overline{k}'')\,) \cdot kc(K(\overline{k}'') \setminus \widehat{\varsigma})(\,\sigma \mid \varsigma \Rightarrow K(\overline{k}'')\,)) \\
& \text{[Corollary 2(a), '}\cup\text{' is nonexpansive]} \\
&= max\{d(\mathcal{D}(s_1 \parallel s_2)(\alpha, K(k))(\sigma), \\
&\qquad\qquad \mathcal{D}(s_1)(\alpha\cdot 1, [\,K(k) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma) \cup \\
&\qquad\qquad \mathcal{D}(s_2)(\alpha\cdot 2, [\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1)\,])(\sigma))\ ^{(5.3.3)}, \\
&\qquad d(\textstyle\bigcup_{\{\alpha'\}\in\Omega_A(K(k'')),\alpha'\neq\alpha} K(k'')(\alpha')(\alpha', K(k'') \setminus \{\alpha'\})(\sigma), \\
&\qquad\qquad \textstyle\bigcup_{\{\alpha'\}\in\Omega_A(K(\overline{k}'')),\alpha'\neq\alpha\cdot 1,\alpha'\neq\alpha\cdot 2} K(\overline{k}'')(\alpha')(\alpha', K(\overline{k}'') \setminus \{\alpha'\})(\sigma))\ ^{(5.3.4)}
\end{aligned}
$$

$$d(\bigcup_{\varsigma\in\Omega_S(K(k''))}(\,\sigma \mid \varsigma \Rightarrow K(k'')\,)\cdot kc(K(k'')\setminus\widehat{\varsigma})(\,\sigma \mid \varsigma \Rightarrow K(k'')\,),$$

$$\bigcup_{\varsigma\in\Omega_S(K(\overline{k}''))}(\,\sigma \mid \varsigma \Rightarrow K(\overline{k}'')\,)\cdot kc(K(\overline{k}'')\setminus\widehat{\varsigma})(\,\sigma \mid \varsigma \Rightarrow K(\overline{k}'')\,))\ ^{(5.3.5)}\ \}$$

From the definition of the denotational semantics function $\mathcal{D}$ it follows immediately that $^{(5.3.3)} = 0$. Next, we treat $^{(5.3.4)}$ and $^{(5.3.5)}$, assuming that:

- $\Omega_A(K(k''))\setminus\{\{\alpha\}\} = \Omega_A(K(\overline{k}''))\setminus\{\{\alpha\cdot 1\},\{\alpha\cdot 2\}\} \neq \emptyset$, and
- $\Omega_S(K(k'')) = \Omega_S(K(\overline{k}'')) \neq \emptyset$.

Indeed, as $(\alpha, k) \in CRes, \alpha\cdot 1 > \alpha, \alpha\cdot 2 > \alpha$ we have:

$$\Omega_S(\kappa) = \Omega_S(K(k'')) \quad \text{[Lemma 5.6(c)]}$$

$$= \omega_S(k'') = \{\{\alpha_1,\cdots,\alpha_m\} \mid \{\alpha_1,\cdots,\alpha_m\}\in\omega_S(k)\wedge$$

$$(\forall 1\leq i\leq m : Q(\alpha_i,\alpha))\}, \text{ and}$$

$$\Omega_S(\overline{\kappa}) = \Omega_S(K(\overline{k}'')) \quad \text{[Lemma 5.6(c)]}$$

$$= \omega_S(\overline{k}'') = \{\{\alpha_1,\cdots,\alpha_m\} \mid \{\alpha_1,\cdots,\alpha_m\}\in\omega_S(k)\wedge$$

$$(\forall 1\leq i\leq m : Q(\alpha_i,\alpha))\}$$

hence $\Omega_S(\kappa) = \omega_S(k'') = \omega_S(\overline{k}'') = \Omega_S(\overline{\kappa}) = \Omega_S(K(k'')) = \Omega_S(K(\overline{k}''))$. As $(\alpha, k) \in CRes, \alpha\cdot 1 > \alpha, \alpha\cdot 2 > \alpha$, we also have

$$\Omega_A(\kappa) = \Omega_A(K(k'')) \quad \text{[Lemma 5.6(b)]}$$

$$= \omega_A(k'') = \{\{\alpha'\} \mid \{\alpha'\}\in\omega_A(k)\wedge Q(\alpha',\alpha)\}\cup\{\{\alpha\}\}, \text{ and}$$

$$\Omega_A(\overline{\kappa}) = \Omega_A(K(\overline{k}'')) \quad \text{[Lemma 5.6(b)]}$$

$$= \omega_A(\overline{k}'') = \{\{\alpha'\} \mid \{\alpha'\}\in\omega_A(k)\wedge Q(\alpha',\alpha)\}\cup\{\{\alpha\cdot 1\},\{\alpha\cdot 2\}\}$$

hence $\omega_A(k'')\setminus\{\{\alpha\}\} = \omega_A(\overline{k}'')\setminus\{\{\alpha\cdot 1\},\{\alpha\cdot 2\}\}$.

Therefore, we can handle $^{(5.3.4)}$ and $^{(5.3.5)}$ as follows:

$$^{(5.3.4)} = d(\bigcup_{\{\alpha'\}\in\Omega_A(K(k'')),\alpha'\neq\alpha} K(k'')(\alpha')(\alpha', K(k'')\setminus\{\alpha'\})(\sigma),$$

$$\bigcup_{\{\alpha'\}\in\Omega_A(K(\overline{k}'')),\alpha'\neq\alpha\cdot 1,\alpha'\neq\alpha\cdot 2} K(\overline{k}'')(\alpha')(\alpha', K(\overline{k}'')\setminus\{\alpha'\})(\sigma))$$

$$= d(\bigcup_{\{\alpha'\}\in\omega_A(k),Q(\alpha',\alpha)} K(k'')(\alpha')(\alpha', K(k'')\setminus\{\alpha'\})(\sigma),$$

$$\bigcup_{\{\alpha'\}\in\omega_A(k),Q(\alpha',\alpha)} K(\overline{k}'')(\alpha')(\alpha', K(\overline{k}'')\setminus\{\alpha'\})(\sigma))$$

$$[\{\alpha'\} \in \omega_A(k) \wedge Q(\alpha',\alpha) \Rightarrow K(k'')(\alpha') = K(\overline{k}'')(\alpha') = K(k)(\alpha'),$$

'$\cup$' is nonexpansive]

$$= max\{d(K(k)(\alpha')(\alpha', (K[\, k \mid \alpha \mapsto (s_1 \parallel s_2)\,]) \setminus \{\alpha'\})(\sigma),$$

$$K(k)(\alpha')(\alpha', (K[\, k \mid \alpha \cdot 1 \mapsto s_1 \mid \alpha \cdot 2 \mapsto s_2\,]) \setminus \{\alpha'\})(\sigma))$$

$$\mid \{\alpha'\} \in \omega_A(k), Q(\alpha',\alpha)\}$$

$[\{\alpha'\} \in \omega_A(k), (\alpha, k) \in CRes, Q(\alpha',\alpha) \Rightarrow$

$\alpha' \neq \alpha, \alpha' \neq \alpha \cdot 1, \alpha' \neq \alpha \cdot 2$; Lemma 5.1, Lemma 5.6(h), Lemma 5.6(g)]

$$= max\{d(\mathcal{D}(s')(\alpha', [\, K(k \setminus \{\alpha'\}) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma),$$

$$\mathcal{D}(s')(\alpha', [\, K(k \setminus \{\alpha'\}) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma))$$

$$\mid \{\alpha'\} \in \omega_A(k), Q(\alpha',\alpha), s' = k(\alpha')\} \; {}^{(5.3.4')}$$

As $(\alpha, k) \in CRes$, it is easy to check that $\forall\{\alpha'\} \in \omega_A(k)$ with $Q(\alpha',\alpha)$ we have $P_{\parallel}(\alpha', \alpha, k \setminus \{\alpha'\})$. Therefore, by taking the maximum of ${}^{(5.3.4')}$ we obtain a result needed in the proof of ${}^{(5.3.1)}$, namely we infer that $\exists s'_1 \in Stat, \sigma'_1 \in \Sigma, \alpha'_1 \in Id$, $k'_1 \in KRes$, $(k'_1 = k \setminus \{\alpha'_1\})$ satisfying $P_{\parallel}(\alpha'_1, \alpha, k'_1)$ such that:

$$d(kc[\, K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,](\sigma),$$

$$kc[\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,](\sigma))$$

$$\leq d(\mathcal{D}(s'_1)(\alpha'_1, [\, K(k'_1) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma'_1),$$

$$\mathcal{D}(s'_1)(\alpha'_1, [\, K(k'_1) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma'_1)) \; {}^{(5.3.4'')}$$

In order to prove ${}^{(5.3.1)}$ we proceed by induction on $|I(k)|$[7] where:

$$I(k) = \{\alpha \mid \alpha \in id(k), k(\alpha) \in J^{\diamondsuit}\}$$

If $|I(k)| = 0$ then by using ${}^{(5.3.4'')}$ we obtain immediatelly the desired result ${}^{(5.3.1)}$, because when $|I(k)| = 0$ we have $\omega_S(k) = \emptyset$ which implies also $\omega_S(k'') = \emptyset$ and $\omega_S(\overline{k}'') = \emptyset$.

Next we treat the case when $|I(k)| > 0$. $\omega(k) = \omega_A(k) \cup \omega_S(k)$ and, in this case at least one of $\omega_A(k)$ or $\omega_S(k)$ is nonempty. Also, by Lemma 5.6(b) and Lemma 5.6(c), $\omega_A(k) = \Omega_A(K(k))$ and $\omega_S(k) = \Omega_S(K(k))$. If ${}^{(5.3.4)} \geq {}^{(5.3.5)}$ then we can proceed as in the basic case (i.e. $|I(k)| = 0$) and we obtain the desired result ${}^{(5.3.1)}$. If ${}^{(5.3.4)} < {}^{(5.3.5)}$ then we compute as follows:

$${}^{(5.3.5)} = d(\textstyle\bigcup_{\varsigma \in \Omega_S(K(k''))} (\, \sigma \mid \varsigma \Rightarrow K(k'')\,) \cdot kc(K(k'') \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow K(k'')\,),$$

[7]Recall that $|\cdot|$ is the cardinal number of '$\cdot$'.

$$\bigcup_{\varsigma\in\Omega_S(K(\overline{k}''))}(\,\sigma \mid \varsigma \Rightarrow K(\overline{k}'')\,)\cdot kc(K(\overline{k}'')\setminus\widehat{\varsigma})(\,\sigma \mid \varsigma \Rightarrow K(\overline{k}'')\,))$$

['$\cup$' is nonexpansive, $\Omega_S(K(k''))=\Omega_S(K(\overline{k}''))=\omega_S(k'')=\omega_S(\overline{k}'')$

$=\{\{\alpha_1,\cdots,\alpha_m\}\mid\{\alpha_1,\cdots,\alpha_m\}\in\omega_S(k)\wedge(\forall 1\leq i\leq m:Q(\alpha_i,\alpha))\}$,

Lemma 5.6(j)]

$$=max\{d((\,\sigma\mid\varsigma\to k''\,)\cdot kc(K(k'')\setminus\widehat{\varsigma})(\,\sigma\mid\varsigma\to k''\,),$$

$$(\,\sigma\mid\varsigma\to\overline{k}''\,)\cdot kc(K(\overline{k}'')\setminus\widehat{\varsigma})(\,\sigma\mid\varsigma\to\overline{k}''\,))$$

$$\mid\varsigma\in\{\{\alpha_1,\cdots,\alpha_m\}\mid\{\alpha_1,\cdots,\alpha_m\}\in\omega_S(k)\wedge$$

$$(\forall 1\leq i\leq m:Q(\alpha_i,\alpha))\}\}\ ^{(5.3.5')}$$

As $(\alpha,k)\in CRes, k''=[\,k\mid\alpha\mapsto s_1\parallel s_2\,]$, $\overline{k}''=[\,k\mid\alpha\cdot 1\mapsto s_1\mid\alpha\cdot 2\mapsto s_2\,]$, for any $\varsigma\in\{\{\alpha_1,\cdots,\alpha_m\}\mid\{\alpha_1,\cdots,\alpha_m\}\in\omega_S(k),(\forall 1\leq i\leq m:Q(\alpha_i,\alpha))\}$ we have:

$$\sigma_{\varsigma k}\overset{not.}{=}(\,\sigma\mid\varsigma\to k''\,)=(\,\sigma\mid\varsigma\to\overline{k}''\,)=(\,\sigma\mid\varsigma\to k\,)$$

because $\forall\{\alpha_1,\cdots,\alpha_m\}\in\omega_S(k)$ such that $\forall 1\leq i\leq m:Q(\alpha_i,\alpha)$, we have: $k''(\alpha_i)=\overline{k}''(\alpha_i)=k(\alpha_i),\quad\forall 1\leq i\leq m$. Hence:

$$\overset{(5.3.5')}{=}max\{d(\sigma_{\varsigma k}\cdot kc((K[\,k\mid\alpha\mapsto s_1\parallel s_2\,])\setminus\{\alpha',\alpha'_1,\cdots,\alpha'_n\})(\sigma_{\varsigma k}),$$

$$\sigma_{\varsigma k}\cdot kc((K[\,k\mid\alpha\cdot 1\mapsto s_1\mid\alpha\cdot 2\mapsto s_2\,])\setminus\{\alpha',\alpha'_1,\cdots,\alpha'_n\})(\sigma_{\varsigma k}))$$

$$\mid\{\alpha',\alpha'_1,\cdots,\alpha'_n\}\in\omega_S(k),$$

$$(Q(\alpha',\alpha)\wedge(\forall 1\leq i\leq n:Q(\alpha'_i,\alpha))),$$

$$k(\alpha')=c_1?v_1\,\&\,\cdots\,\&\,c_n?v_n,$$

$$k(\alpha'_1)=c_1?\xi_1,\cdots,k(\alpha'_n)=c_n!\xi_n,$$

$$\sigma_{\varsigma k}=[\,\sigma\mid v_1\mapsto\xi_1(\sigma)\mid\cdots\mid v_n\mapsto\xi_n(\sigma)\,]\}$$

$[\{\alpha',\alpha'_1,\cdots,\alpha'_n\}\in\omega_S(k),(\alpha,k)\in CRes\Rightarrow$

$\alpha\notin\{\alpha',\alpha'_1,\cdots,\alpha'_n\},\alpha\cdot 1\notin\{\alpha',\alpha'_1,\cdots,\alpha'_n\}$,

$\alpha\cdot 2\notin\{\alpha',\alpha'_1,\cdots,\alpha'_n\}$; Lemma 5.1, Lemma 5.6(h), Lemma 5.6(g)]

$$=\tfrac{1}{2}\cdot max\{d(kc[\,K(k\setminus\{\alpha',\alpha'_1,\cdots,\alpha'_n\})\mid\alpha\mapsto\mathcal{D}(s_1\parallel s_2)\,](\sigma_{\varsigma k}),$$

$$kc[\,K(k\setminus\{\alpha',\alpha'_1,\cdots,\alpha'_n\})\mid\alpha\cdot 1\mapsto\mathcal{D}(s_1)\mid\alpha\cdot 2\mapsto\mathcal{D}(s_2)\,](\sigma_{\varsigma k}))$$

$$\mid\{\alpha',\alpha'_1,\cdots,\alpha'_n\}\in\omega_S(k),$$

$$(Q(\alpha',\alpha) \wedge (\forall 1 \le i \le n : Q(\alpha'_i,\alpha))),$$

$$k(\alpha') = c_1?v_1 \,\&\, \cdots \,\&\, c_n?v_n,$$

$$k(\alpha'_1) = c_1?\xi_1, \cdots, k(\alpha'_n) = c_n!\xi_n,$$

$$\sigma_{\varsigma k} = [\,\sigma \mid v_1 \mapsto \xi_1(\sigma) \mid \cdots \mid v_n \mapsto \xi_n(\sigma)\,]\} \;^{(5.3.5'')}$$

Notice that $\forall\{\alpha',\alpha'_1,\cdots,\alpha'_n\} \in \omega_S(k) : |I(k\backslash\{\alpha',\alpha'_1,\cdots,\alpha'_n\})| < |I(k)|$ and $(\alpha, k\backslash\{\alpha',\alpha'_1,\cdots,\alpha'_n\}) \in CRes$. Therefore we can use the induction hypothesis for $^{(5.3.1)}$ and we infer that $\exists \overline{s}' \in Stat, \overline{\alpha}' \in Id, \overline{\sigma}' \in \Sigma, \overline{k}' \in KRes$ such that $P_{\|}(\overline{\alpha}',\alpha,\overline{k}')$ and

$$\begin{aligned}
&d(kc[\,K(k \setminus \{\alpha',\alpha'_1,\cdots,\alpha'_n\}) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,](\sigma_{\varsigma k}),\\
&\quad kc[\,K(k \setminus \{\alpha',\alpha'_1,\cdots,\alpha'_n\}) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_2)\,](\sigma_{\varsigma k}))\\
&\le d(\mathcal{D}(\overline{s}')(\overline{\alpha}',[\,K(\overline{k}') \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\overline{\sigma}'),\\
&\qquad \mathcal{D}(\overline{s}')(\overline{\alpha}',[\,K(\overline{k}') \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_2)\,])(\overline{\sigma}'))
\end{aligned}$$

for any $\varsigma \in \omega_S(k), \varsigma = \{\alpha',\alpha'_1,\cdots,\alpha'_n\}, Q(\alpha',\alpha), Q(\alpha'_i,\alpha), \forall 1 \le i \le n$. By taking the maximum over $^{(5.3.5'')}$ it follows immediately that $\exists \overline{s}'_1 \in Stat, \overline{\alpha}'_1 \in Id, \overline{\sigma}'_1 \in \Sigma, \overline{k}'_1 \in KRes$ such that $P_{\|}(\overline{\alpha}'_1,\alpha,\overline{k}'_1)$ and:

$$\begin{aligned}
&^{(5.3.5)} \;=\; ^{(5.3.5'')}\\
&\le d(\mathcal{D}(\overline{s}'_1)(\overline{\alpha}'_1,[\,K(\overline{k}'_1) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\overline{\sigma}'_1),\\
&\qquad \mathcal{D}(\overline{s}'_1)(\overline{\alpha}'_1,[\,K(\overline{k}'_1) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_2)\,])(\overline{\sigma}'_1))
\end{aligned}$$

This implies that we have

$$\begin{aligned}
&d(kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,](\sigma),\\
&\quad kc[\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_2)\,](\sigma))\\
&\le d(\mathcal{D}(\overline{s}'_1)(\overline{\alpha}'_1,[\,K(\overline{k}'_1) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\overline{\sigma}'_1),\\
&\qquad \mathcal{D}(\overline{s}'_1)(\overline{\alpha}'_1,[\,K(\overline{k}'_1) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_2)\,])(\overline{\sigma}'_1))
\end{aligned}$$

which completes the proof of $^{(5.3.1)}$, for any $k \in KRes$, $|I(k)| \ge 0$.

Next, we prove $^{(5.3.2)}$. We proceed by induction on $c(s_0)$ using $^{(5.3.1)}$. In the computations given below, by assumption $P_{\|}(\alpha_0,\alpha,k)$.

Case $[s_0 = \mathsf{skip}]$. Simple enough; left to the reader.

Case $[s_0 = v := e]$. Let $\overline{\sigma}' = [\,\sigma \mid v \mapsto \mathcal{E}[\![e]\!](\sigma)\,]$.

$$d(\mathcal{D}(v := e)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma),$$
$$\mathcal{D}(v := e)(\alpha_0, [\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma))$$
$$= d(\overline{\sigma}' \cdot kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,](\overline{\sigma}'),$$
$$\overline{\sigma}' \cdot kc[\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,](\overline{\sigma}'))$$
$$= \tfrac{1}{2} \cdot\ d(kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,](\overline{\sigma}'),$$
$$kc[\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,](\overline{\sigma}'))\ ^{(5.3.8)}$$

As $(\alpha, k) \in CRes$, by $^{(5.3.1)}$, $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ such that $P_{\parallel}(\alpha', \alpha, k')$ and:

$$^{(5.3.8)}\ \leq \tfrac{1}{2} \cdot\ d(\mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma'),$$
$$\mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma'))$$

Case $[s_0 = j]$.

$$d(\mathcal{D}(j)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma),$$
$$\mathcal{D}(j)(\alpha_0, [\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma))$$
$$= d(\sigma \cdot kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2) \mid \alpha_0 \mapsto \langle j \rangle\,](\sigma),$$
$$\sigma \cdot kc[\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2) \mid \alpha_0 \mapsto \langle j \rangle\,](\sigma))$$

[$P_{\parallel}(\alpha_0, \alpha, k) \Rightarrow \alpha_0 \neq \alpha, \alpha_0 \neq \alpha \cdot 1, \alpha_0 \neq \alpha \cdot 2$; Lemma 5.1]

$$= \tfrac{1}{2} \cdot\ d(kc[\,K[\,k \mid \alpha_0 \mapsto \langle j \rangle\,] \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,](\sigma),$$
$$kc[\,K[\,k \mid \alpha_0 \mapsto \langle j \rangle\,] \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,](\sigma))\ ^{(5.3.9)}$$

Notice that $P_{\parallel}(\alpha_0, \alpha, k)$ implies $(\alpha, [\,k \mid \alpha_0 \mapsto \langle j \rangle\,]) \in CRes$, therefore, by $^{(5.3.1)}$ $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ such that $P_{\parallel}(\alpha', \alpha, k')$ and:

$$^{(5.3.9)}\ \leq \tfrac{1}{2} \cdot\ d(\mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma'),$$
$$\mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma'))$$

Case $[s_0 = c!e]$. Similar to the case $s_0 = j$.

Case $[s_0 = x]$.

$$d(\mathcal{D}(x)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma),$$
$$\mathcal{D}(x)(\alpha_0, [\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma))$$
$$= d(\mathcal{D}(D(x))(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma),$$
$$\mathcal{D}(D(x))(\alpha_0, [\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma))\ ^{(5.3.10)}$$

By the induction hypothesis $(c(D(x)) < c(x))$ $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ such that $P_{\parallel}(\alpha', \alpha, k')$ and:

$$^{(5.3.10)}\ \leq \tfrac{1}{2} \cdot\ d(\mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma'),$$

$$\mathcal{D}(s')(\alpha', [\, K(k') \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma'))$$

Case $[s_0 = s_0^1 + s_0^2]$. It has any difficulty, and so we skip it.

Case $[s_0 = s_0^1; s_0^2]$.

$$\begin{aligned}
& d(\mathcal{D}(s_0^1; s_0^2)(\alpha_0, [\, K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma), \\
& \quad \mathcal{D}(s_0^1; s_0^2)(\alpha_0, [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma)) \\
& = d(\mathcal{D}(s_0^1)(\alpha_0 {\cdot} 1, [\, K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2) \mid \alpha_0 \mapsto \mathcal{D}(s_0^2)\,])(\sigma), \\
& \qquad \mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2) \mid \alpha_0 \mapsto \mathcal{D}(s_0^2)\,])(\sigma)) \\
& [\ P_{\parallel}(\alpha_0, \alpha, k) \Rightarrow \alpha_0 \neq \alpha, \alpha_0 \neq \alpha \cdot 1, \alpha_0 \neq \alpha \cdot 2; \text{Lemma 5.1}\ ] \\
& = d(\mathcal{D}(s_0^1)(\alpha_0 {\cdot} 1, [\, K[\, k \mid \alpha_0 \mapsto s_0^2\,] \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma), \\
& \qquad \mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\, K[\, k \mid \alpha_0 \mapsto s_0^2\,] \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma)) \ ^{(5.3.11)}
\end{aligned}$$

Notice that $P_{\parallel}(\alpha_0, \alpha, k)$ implies $P_{\parallel}(\alpha_0 \cdot 1, \alpha, [\, k \mid \alpha_0 \mapsto s_0^2\,])$. Therefore, we can use the induction hypothesis ($c(s_0^1) < c(s_0^1; s_0^2)$) and we infer that $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ such that $P_{\parallel}(\alpha', \alpha, k')$ and

$$\begin{aligned}
{}^{(5.3.11)} \quad & \leq \tfrac{1}{2} \cdot\ d(\mathcal{D}(s')(\alpha', [\, K(k') \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma'), \\
& \qquad \mathcal{D}(s')(\alpha', [\, K(k') \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma'))
\end{aligned}$$

Case $[s_0 = s_0^1 \| s_0^2]$.

$$\begin{aligned}
& d(\mathcal{D}(s_0^1 \| s_0^2)(\alpha_0, [\, K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma), \\
& \quad \mathcal{D}(s_0^1 \| s_0^2)(\alpha_0, [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma)) \\
& = d(\mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\, K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2) \mid \alpha_0 \cdot 2 \mapsto \mathcal{D}(s_0^2)\,])(\sigma) \cup \\
& \qquad \mathcal{D}(s_0^2)(\alpha_0 \cdot 2, [\, K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2) \mid \alpha_0 \cdot 1 \mapsto \mathcal{D}(s_0^1)\,])(\sigma), \\
& \qquad \mathcal{D}(s_0^1)(\alpha_0 \cdot 1, \\
& \qquad\qquad [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2) \mid \alpha_0 \cdot 2 \mapsto \mathcal{D}(s_0^2)\,])(\sigma) \cup \\
& \qquad \mathcal{D}(s_0^2)(\alpha_0 \cdot 2, \\
& \qquad\qquad [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2) \mid \alpha_0 \cdot 1 \mapsto \mathcal{D}(s_0^1)\,])(\sigma)) \\
& [\ \text{'}\cup\text{' is nonexpansive}\ ] \\
& \leq max\{d(\mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\, K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2) \mid \alpha_0 \cdot 2 \mapsto \mathcal{D}(s_0^2)\,])(\sigma), \\
& \qquad\quad \mathcal{D}(s_0^1)(\alpha_0 \cdot 1, \\
& \qquad\qquad\quad [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2) \mid \alpha_0 \cdot 2 \mapsto \mathcal{D}(s_0^2)\,])(\sigma)), \\
& \qquad d(\mathcal{D}(s_0^2)(\alpha_0 \cdot 2, [\, K(k) \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2) \mid \alpha_0 \cdot 1 \mapsto \mathcal{D}(s_0^1)\,])(\sigma), \\
& \qquad\quad \mathcal{D}(s_0^2)(\alpha_0 \cdot 2, \\
& \qquad\qquad\quad [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2) \mid \alpha_0 \cdot 1 \mapsto \mathcal{D}(s_0^1)\,])(\sigma))\}
\end{aligned}$$

$[\ P_{\parallel}(\alpha_0, \alpha, k) \Rightarrow \alpha_0 \cdot 2 \notin \{\alpha, \alpha \cdot 1, \alpha \cdot 2\}, \alpha_0 \cdot 1 \notin \{\alpha, \alpha \cdot 1, \alpha \cdot 2\};$ Lemma 5.1$]$

$$= max\{d(\mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\,K[\,k \mid \alpha_0 \cdot 2 \mapsto s_0^2\,] \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma),$$
$$\mathcal{D}(s_0^1)(\alpha_0 \cdot 1,$$
$$[\,K[\,k \mid \alpha_0 \cdot 2 \mapsto s_0^2\,] \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma)), \ ^{(5.3.12)}$$
$$d(\mathcal{D}(s_0^2)(\alpha_0 \cdot 2, [\,K[\,k \mid \alpha_0 \cdot 1 \mapsto s_0^1\,] \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma),$$
$$\mathcal{D}(s_0^2)(\alpha_0 \cdot 2,$$
$$[\,K[\,k \mid \alpha_0 \cdot 1 \mapsto s_0^1\,] \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma)) \ ^{(5.3.13)}\ \}$$

$P_{\parallel}(\alpha_0, \alpha, k)$ implies $P_{\parallel}(\alpha_0 \cdot 1, \alpha, [\,k \mid \alpha_0 \cdot 2 \mapsto s_0^2\,])$ and $P_{\parallel}(\alpha_0 \cdot 2, \alpha, [\,k \mid \alpha_0 \cdot 1 \mapsto s_0^1\,])$. Therefore, we can use the induction hypothesis (because $c(s_0^1) < c(s_0^1 \parallel s_0^2)$ and $c(s_0^2) < c(s_0^1 \parallel s_0^2)$) and we infer that (1) $\exists s_1' \in Stat, \sigma_1' \in \Sigma, \alpha_1' \in Id, k_1' \in KRes$ such that $P_{\parallel}(\alpha_1', \alpha, k_1')$ and (2) $\exists s_2' \in Stat, \sigma_2' \in \Sigma, \alpha_2' \in Id, k_2' \in KRes$ such that $P_{\parallel}(\alpha_2', \alpha, k_2')$ and:

$$^{(5.3.12)} \leq \tfrac{1}{2} \cdot\ d(\mathcal{D}(s_1')(\alpha_1', [\,K(k_1') \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma_1'),$$
$$\mathcal{D}(s_1')(\alpha_1', [\,K(k_1') \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma_1')) \ ^{(5.3.12')}$$
$$^{(5.3.13)} \leq \tfrac{1}{2} \cdot\ d(\mathcal{D}(s_2')(\alpha_2', [\,K(k_2') \mid \alpha \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma_2'),$$
$$\mathcal{D}(s_2')(\alpha_2', [\,K(k_2') \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma_2')) \ ^{(5.3.13')}$$

Therefore, by taking $max\{\ ^{(5.3.12')}\ ,\ ^{(5.3.13')}\ \}$ we obtain the desired result $\ ^{(5.3.2)}$.
□

LEMMA 5.4 *For all* $s_0, s_1, s_2 \in Stat, \sigma \in \Sigma, \alpha_0, \alpha \in Id, k \in KRes$ *such that* $(\alpha_0, k) \in CRes, \alpha \notin id(k), \alpha \cdot 1 \notin id(k)$ *and* $(\neg(\alpha_0 \leq \alpha \cdot 1))$ *we have:*

(a) $kc([\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1; s_2)\,])(\sigma) =$

$kc([\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma)$

(b) $\mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1; s_2)\,])(\sigma) =$

$\mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma)$

Proof: The proof of this Lemma is similar to the proof of Lemma 5.3. In this case the invariant property which is preserved by the computation steps is $P_;$:

$$P_;(\alpha_0, \alpha, k) \overset{not.}{=} ((\alpha_0, k) \in CRes) \wedge (\alpha \notin id(k)) \wedge (\alpha \cdot 1 \notin id(k)) \wedge (\neg(\alpha_0 \leq \alpha \cdot 1))$$

We proceed as follows. For 5.4(a) we show that $\forall s_1, s_2 \in Stat, \sigma \in \Sigma, \alpha \in Id, k \in KRes$ such that $\alpha \notin id(k)$ and $\alpha \cdot 1 \notin id(k)$, $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ such that $P_;(\alpha', \alpha, k')$ such that:

$$\epsilon_{kc}^{;}(k, \alpha, s_1, s_2, \sigma) \overset{not.}{=}$$

$$\begin{aligned}
& d(kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1; s_2)\,](\sigma), \\
& \quad kc[\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,](\sigma)) \\
\leq\ & d(\mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1; s_2)\,])(\sigma'), \\
& \quad \mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma'))\ ^{(5.4.1)} \\
\stackrel{not.}{=}\ & \epsilon^{;}_{\mathcal{D}}(s', \alpha', k', \alpha, s_1, s_2, \sigma')
\end{aligned}$$

Also, for 5.4(b) we show that $\forall s_0, s_1, s_2 \in Stat, \sigma \in \Sigma, \alpha_0, \alpha \in Id, k \in KRes$ such that $P_;(\alpha_0, \alpha, k)$, $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ such that $P_;(\alpha', \alpha, k')$ and:

$$\begin{aligned}
& \epsilon^{;}_{\mathcal{D}}(s_0, \alpha_0, k, \alpha, s_1, s_2, \sigma) \\
=\ & d(\mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1; s_2)\,])(\sigma), \\
& \quad \mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma)) \\
\leq\ & \tfrac{1}{2} \cdot d(\mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1; s_2)\,])(\sigma'), \\
& \quad\quad \mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma'))\ \ ^{(5.4.2)} \\
\stackrel{not.}{=}\ & \epsilon^{;}_{\mathcal{D}}(s', \alpha', k', \alpha, s_1, s_2, \sigma')
\end{aligned}$$

If we put $(w \in)W = Stat \times Id \times KRes \times Id \times Stat \times Stat \times \Sigma$ we infer that:

$$sup_{\ w \in W \,:\, P_;(\alpha_0, \alpha, k)}\ \epsilon^{;}_{\mathcal{D}}(w) \leq \tfrac{1}{2} \cdot sup_{\ w' \in W \,:\, P_;(\alpha', \alpha, k')}\ \epsilon^{;}_{\mathcal{D}}(w')$$

where $w = (s_0, \alpha_0, k, \alpha, s_1, s_2, \sigma)$ and $w' = (s', \alpha', k', \alpha, s_1, s_2, \sigma')$.
However $sup_{\ w \in W \,:\, P_;(\alpha_0, \alpha, k)}\ \epsilon^{;}_{\mathcal{D}}(w) = sup_{\ w' \in W \,:\, P_;(\alpha', \alpha, k')}\ \epsilon^{;}_{\mathcal{D}}(w')$. So we infer that:

$$sup_{\ w \in W \,:\, P_;(\alpha_0, \alpha, k)}\ \epsilon^{;}_{\mathcal{D}}(w) = 0$$

which implies immediately 5.4(b). Next, by using this result and $^{(5.4.1)}$ we obtain immediately 5.4(a).

As in the case of property $^{(5.3.1)}$ (used in the proof of Lemma 5.3(a)), in the proof of $^{(5.4.1)}$ one can proceed by induction on $|I(k)|$, where $I(k) = \{\alpha \mid \alpha \in id(k), k(\alpha) \in J^{\diamond}\}$. Here we only give the proof of $^{(5.4.2)}$. We proceed by induction on $c(s_0)$, using $^{(5.4.1)}$. In the computations given below, we assume $P_;(\alpha_0, \alpha, k)$, and have three subcases.

Case $[s_0 = j]$.

$$\begin{aligned}
& d(\mathcal{D}(j)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1; s_2)\,])(\sigma), \\
& \quad \mathcal{D}(j)(\alpha_0, [\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma)) \\
=\ & d(\sigma \cdot kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1; s_2) \mid \alpha_0 \mapsto \langle j \rangle\,](\sigma), \\
& \quad\quad \sigma \cdot kc[\,K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2) \mid \alpha_0 \mapsto \langle j \rangle\,](\sigma))
\end{aligned}$$

$[\ P_;(\alpha_0,\alpha,k) \Rightarrow \alpha_0 \neq \alpha, \alpha_0 \neq \alpha\cdot 1$; Lemma 5.1 $]$

$$= \tfrac{1}{2}\cdot d(kc[\,K[\,k \mid \alpha_0 \mapsto \langle j\rangle\,] \mid \alpha \mapsto \mathcal{D}(s_1;s_2)\,](\sigma),$$
$$kc[\,K[\,k \mid \alpha_0 \mapsto \langle j\rangle\,] \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,](\sigma))\ ^{(5.4.3)}$$

$P_;(\alpha_0,\alpha,k)$ implies $\alpha \notin id([\,k \mid \alpha_0 \mapsto \langle j\rangle\,]), \alpha\cdot 1 \notin id([\,k \mid \alpha_0 \mapsto \langle j\rangle\,])$. Therefore, by $^{(5.4.1)}$ $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ such that $P_;(\alpha',\alpha,k')$ and:

$$^{(5.4.3)} \leq \tfrac{1}{2}\cdot d(\mathcal{D}(s')(\alpha',[\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1;s_2)\,])(\sigma'),$$
$$\mathcal{D}(s')(\alpha',[\,K(k') \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma'))$$

Case $[s_0 = x]$.

$$d(\mathcal{D}(x)(\alpha_0,[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1;s_2)\,])(\sigma),$$
$$\mathcal{D}(x)(\alpha_0,[\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma))$$
$$= d(\mathcal{D}(D(x))(\alpha_0,[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1;s_2)\,])(\sigma),$$
$$\mathcal{D}(D(x))(\alpha_0,[\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma))\ ^{(5.4.4)}$$

By the induction hypothesis $(c(D(x)) < c(x))$ $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ such that $P_;(\alpha',\alpha,k')$ and:

$$^{(5.4.4)} \leq \tfrac{1}{2}\cdot d(\mathcal{D}(s')(\alpha',[\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1;s_2)\,])(\sigma'),$$
$$\mathcal{D}(s')(\alpha',[\,K(k') \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma'))$$

Case $[s_0 = s_0^1; s_0^2]$.

$$d(\mathcal{D}(s_0^1;s_0^2)(\alpha_0,[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1;s_2)\,])(\sigma),$$
$$\mathcal{D}(s_0^1;s_0^2)(\alpha_0,[\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma))$$
$$= d(\mathcal{D}(s_0^1)(\alpha_0\cdot 1,[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1;s_2) \mid \alpha_0 \mapsto \mathcal{D}(s_0^2)\,])(\sigma),$$
$$\mathcal{D}(s_0^1)(\alpha_0\cdot 1,[\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2) \mid \alpha_0 \mapsto \mathcal{D}(s_0^2)\,])(\sigma))$$

$[\ P_;(\alpha_0,\alpha,k) \Rightarrow \alpha_0 \neq \alpha, \alpha_0 \neq \alpha\cdot 1$; Lemma 5.1 $]$

$$= d(\mathcal{D}(s_0^1)(\alpha_0\cdot 1,[\,K[\,k \mid \alpha_0 \mapsto s_0^2\,] \mid \alpha \mapsto \mathcal{D}(s_1;s_2)\,])(\sigma),$$
$$\mathcal{D}(s_0^1)(\alpha_0\cdot 1,[\,K[\,k \mid \alpha_0 \mapsto s_0^2\,] \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma))\ ^{(5.4.5)}$$

$P_;(\alpha_0,\alpha,k)$ implies $P_;(\alpha_0\cdot 1,\alpha,[\,k \mid \alpha_0 \mapsto s_0^2\,])$. Therefore, we can use the induction hypothesis $(c(s_0^1) < c(s_0^1;s_0^2))$ and we infer that $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ such that $P_;(\alpha',\alpha,k')$ and

$$^{(5.4.5)} \leq \tfrac{1}{2}\cdot d(\mathcal{D}(s')(\alpha',[\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1;s_2)\,])(\sigma'),$$
$$\mathcal{D}(s')(\alpha',[\,K(k') \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma')) \qquad \square$$

LEMMA 5.5 *For all* $s_0, s_1, s_2 \in Stat, \sigma \in \Sigma, \alpha_0, \alpha \in Id, k \in KRes$ *such that* $(\alpha_0, k) \in CRes, \alpha \notin id(k)$ *and* $(\neg(\alpha_0 \leq \alpha))$ *we have:*

(a) $kc([\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1+s_2)\,])(\sigma)$

$$= kc([\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1)\,])(\sigma) \cup kc([\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma)$$

(b) $\mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 + s_2)\,])(\sigma)$

$$= \mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1)\,])(\sigma) \cup \mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma)$$

Proof: First we introduce the invariant property which is preserved by the computation steps.

$$P_{;+}(\alpha_0, \alpha, k) \overset{not.}{=} ((\alpha_0, k) \in CRes) \wedge (\alpha \notin id(k)) \wedge (\neg(\alpha_0 \leq \alpha))$$

For 5.5(a) we show that $\forall s_1, s_2 \in Stat, \sigma \in \Sigma, \alpha \in Id, k \in KRes$ such that $\alpha \notin id(k)$, $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ such that $P_{;+}(\alpha', \alpha, k')$ and:

$$\begin{aligned}
&\epsilon^{;+}_{kc}(k, \alpha, s_1, s_2, \sigma) \overset{not.}{=} \\
&\quad d(kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 + s_2)\,](\sigma), \\
&\qquad kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1)\,](\sigma) \cup kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2)\,](\sigma)) \\
&\leq d(\mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1 + s_2)\,])(\sigma'), \\
&\qquad \mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1)\,])(\sigma') \cup \mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma'))) \quad (5.5.1) \\
&\overset{not.}{=} \epsilon^{;+}_{\mathcal{D}}(s', \alpha', k', \alpha, s_1, s_2, \sigma')
\end{aligned}$$

Also, for 5.5(b) we show that $\forall s_0, s_1, s_2 \in Stat, \sigma \in \Sigma, \alpha_0, \alpha \in Id, k \in KRes$ such that $P_{;+}(\alpha_0, \alpha, k)$, $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ such that $P_{;+}(\alpha', \alpha, k')$ and:

$$\begin{aligned}
&\epsilon^{;+}_{\mathcal{D}}(s_0, \alpha_0, k, \alpha, s_1, s_2, \sigma) \\
&= d(\mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 + s_2)\,])(\sigma), \\
&\qquad \mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1)\,])(\sigma) \cup \mathcal{D}(s_0)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma)) \\
&\leq \tfrac{1}{2} \cdot d(\mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1 + s_2)\,])(\sigma'), \\
&\qquad \mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1)\,])(\sigma') \cup \mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma')) \quad (5.5.2) \\
&= \epsilon^{;+}_{\mathcal{D}}(s', \alpha', k', \alpha, s_1, s_2, \sigma')
\end{aligned}$$

If we put $(w \in)W = Stat \times Id \times KRes \times Id \times Stat \times Stat \times \Sigma$ we infer that:

$$sup_{\,w \in W \,:\, P_{;+}(\alpha_0, \alpha, k)}\; \epsilon^{;+}_{\mathcal{D}}(w) \leq \tfrac{1}{2} \cdot sup_{\,w' \in W \,:\, P_{;+}(\alpha', \alpha, k')}\; \epsilon^{;+}_{\mathcal{D}}(w')$$

where $w = (s_0, \alpha_0, k, \alpha, s_1, s_2, \sigma)$ and $w' = (s', \alpha', k', \alpha, s_1, s_2, \sigma')$.
Obviously, $sup_{\,w \in W \,:\, P_{;+}(\alpha_0, \alpha, k)}\; \epsilon^{;+}_{\mathcal{D}}(w) = sup_{\,w' \in W \,:\, P_{;+}(\alpha', \alpha, k')}\; \epsilon^{;+}_{\mathcal{D}}(w')$. We infer that:

$$sup_{\ w\in W\,:\,P_{;+}(\alpha_0,\alpha,k)}\ \epsilon^{;+}_{\mathcal{D}}(w) = 0$$

which implies 5.5(b). Next, by using this result and and $^{(5.5.1)}$ we obtain 5.5(a).

In the proof of $^{(5.5.1)}$ one can proceed by induction on $|I(k)|$, where $I(k) = \{\alpha \mid \alpha \in id(k), k(\alpha) \in J^{\diamond}\}$. One can follow the same line of reasoning as in the proofs of 5.3(a) ($^{(5.3.1)}$) and 5.4(a) ($^{(5.4.1)}$). Here, we only give the proof of $^{(5.5.2)}$. We proceed by induction on $c(s_0)$, using $^{(5.5.1)}$. In the computations given below, we assume $P_{;+}(\alpha_0, \alpha, k)$. We only treat two subcases.

Case $[s_0 = \mathsf{skip}]$.

$$\begin{aligned}
& d(\mathcal{D}(\,\mathsf{skip}\,)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1+s_2)\,])(\sigma),\\
&\quad \mathcal{D}(\,\mathsf{skip}\,)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1)\,])(\sigma) \cup\\
&\quad \mathcal{D}(\,\mathsf{skip}\,)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma))\\
&= d(\sigma \cdot kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1+s_2)\,](\sigma),\\
&\qquad \sigma \cdot kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1)\,](\sigma) \cup\\
&\qquad \sigma \cdot kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2)\,](\sigma))\\
&= d(\sigma \cdot kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1+s_2)\,](\sigma),\\
&\qquad \sigma \cdot (kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1)\,](\sigma) \cup\\
&\qquad\qquad kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2)\,](\sigma)))\\
&= \tfrac{1}{2} \cdot\ d(kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1+s_2)\,](\sigma),\\
&\qquad\qquad kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1)\,](\sigma) \cup\\
&\qquad\qquad kc[\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2)\,](\sigma))\ ^{(5.5.3)}
\end{aligned}$$

As $P_{;+}(\alpha_0, \alpha, k) \Rightarrow \alpha \notin id(k)$, by $^{(5.5.1)}$ $\exists s' \in Stat, \sigma' \in \Sigma, \alpha' \in Id, k' \in KRes$ such that $P_{;+}(\alpha', \alpha, k')$ and:

$$\begin{aligned}
^{(5.5.3)}\ \ & \le \tfrac{1}{2} \cdot\ d(\mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1+s_2)\,])(\sigma'),\\
&\qquad \mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_1)\,])(\sigma') \cup\\
&\qquad \mathcal{D}(s')(\alpha', [\,K(k') \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma'))
\end{aligned}$$

Case $[s_0 = s_0^1 \| s_0^2]$.

$$\begin{aligned}
& d(\mathcal{D}(s_0^1 \parallel s_0^2)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1+s_2)\,])(\sigma),\\
&\quad \mathcal{D}(s_0^1 \parallel s_0^2)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1)\,])(\sigma) \cup\\
&\quad \mathcal{D}(s_0^1 \parallel s_0^2)(\alpha_0, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma))\\
&= d(\mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1+s_2) \mid \alpha_0 \cdot 2 \mapsto \mathcal{D}(s_0^2)\,])(\sigma) \cup\\
&\qquad \mathcal{D}(s_0^2)(\alpha_0 \cdot 2, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1+s_2) \mid \alpha_0 \cdot 1 \mapsto \mathcal{D}(s_0^1)\,])(\sigma),\\
&\qquad \mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1) \mid \alpha_0 \cdot 2 \mapsto \mathcal{D}(s_0^2)\,])(\sigma) \cup
\end{aligned}$$

$$\mathcal{D}(s_0^2)(\alpha_0 \cdot 2, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1) \mid \alpha_0 \cdot 1 \mapsto \mathcal{D}(s_0^1)\,])(\sigma) \cup$$
$$\mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2) \mid \alpha_0 \cdot 2 \mapsto \mathcal{D}(s_0^2)\,])(\sigma) \cup$$
$$\mathcal{D}(s_0^2)(\alpha_0 \cdot 2, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2) \mid \alpha_0 \cdot 1 \mapsto \mathcal{D}(s_0^1)\,])(\sigma))$$

['$\cup$' is associative, commutative and non-expansive]

$$\le max\{d(\mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 + s_2) \mid \alpha_0 \cdot 2 \mapsto \mathcal{D}(s_0^2)\,])(\sigma),$$
$$\mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1) \mid \alpha_0 \cdot 2 \mapsto \mathcal{D}(s_0^2)\,])(\sigma) \cup$$
$$\mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2) \mid \alpha_0 \cdot 2 \mapsto \mathcal{D}(s_0^2)\,])(\sigma)),$$
$$d(\mathcal{D}(s_0^2)(\alpha_0 \cdot 2, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1 + s_2) \mid \alpha_0 \cdot 1 \mapsto \mathcal{D}(s_0^1)\,])(\sigma),$$
$$\mathcal{D}(s_0^2)(\alpha_0 \cdot 2, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_1) \mid \alpha_0 \cdot 1 \mapsto \mathcal{D}(s_0^1)\,])(\sigma) \cup$$
$$\mathcal{D}(s_0^2)(\alpha_0 \cdot 2, [\,K(k) \mid \alpha \mapsto \mathcal{D}(s_2) \mid \alpha_0 \cdot 1 \mapsto \mathcal{D}(s_0^1)\,])(\sigma))\}$$

[$P_{;+}(\alpha_0, \alpha, k) \Rightarrow \alpha \neq \alpha_0 \cdot 1, \alpha \neq \alpha_0 \cdot 2$; Lemma 5.1]

$$= max\{d(\mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\,K[\,k \mid \alpha_0 \cdot 2 \mapsto s_0^2\,] \mid \alpha \mapsto \mathcal{D}(s_1 + s_2)\,])(\sigma),$$
$$\mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\,K[\,k \mid \alpha_0 \cdot 2 \mapsto s_0^2\,] \mid \alpha \mapsto \mathcal{D}(s_1)\,])(\sigma) \cup$$
$$\mathcal{D}(s_0^1)(\alpha_0 \cdot 1, [\,K[\,k \mid \alpha_0 \cdot 2 \mapsto s_0^2\,] \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma)), \quad ^{(5.5.4)}$$
$$d(\mathcal{D}(s_0^2)(\alpha_0 \cdot 2, [\,K[\,k \mid \alpha_0 \cdot 1 \mapsto s_0^1\,] \mid \alpha \mapsto \mathcal{D}(s_1 + s_2)\,])(\sigma),$$
$$\mathcal{D}(s_0^2)(\alpha_0 \cdot 2, [\,K[\,k \mid \alpha_0 \cdot 1 \mapsto s_0^1\,] \mid \alpha \mapsto \mathcal{D}(s_1)\,])(\sigma) \cup$$
$$\mathcal{D}(s_0^2)(\alpha_0 \cdot 2, [\,K[\,k \mid \alpha_0 \cdot 1 \mapsto s_0^1\,] \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma)) \quad ^{(5.5.5)}$$
$$\}$$

$P_{;+}(\alpha_0, \alpha, k)$ implies $P_{;+}(\alpha_0 \cdot 1, \alpha, [\,k \mid \alpha_0 \cdot 2 \mapsto s_0^2\,])$ and $P_{;+}(\alpha_0 \cdot 1, \alpha, [\,k \mid \alpha_0 \cdot 2 \mapsto s_0^2\,])$. Therefore, by the induction hypothesis ($c(s_0^1) < c(s_0^1 + s_0^2)$ and $c(s_0^2) < c(s_0^1 + s_0^2)$) we infer that (1) $\exists s_1' \in Stat, \sigma_1' \in \Sigma, \alpha_1' \in Id, k_1' \in KRes$ such that $P_{;+}(\alpha_1', \alpha, k_1')$ and (2) $\exists s_2' \in Stat, \sigma_2' \in \Sigma, \alpha_2' \in Id, k_2' \in KRes$ such that $P_{;+}(\alpha_2', \alpha, k_2')$ and:

$$^{(5.5.4)} \quad \le \tfrac{1}{2} \cdot d(\mathcal{D}(s_1')(\alpha_1', [\,K(k_1') \mid \alpha \mapsto \mathcal{D}(s_1 + s_2)\,])(\sigma_1'),$$
$$\mathcal{D}(s_1')(\alpha_1', [\,K(k_1') \mid \alpha \mapsto \mathcal{D}(s_1)\,])(\sigma_1') \cup$$
$$\mathcal{D}(s_1')(\alpha_1', [\,K(k_1') \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma_1')) \quad ^{(5.5.4')}$$
$$^{(5.5.5)} \quad \le \tfrac{1}{2} \cdot d(\mathcal{D}(s_2')(\alpha_2', [\,K(k_2') \mid \alpha \mapsto \mathcal{D}(s_1 + s_2)\,])(\sigma_2'),$$
$$\mathcal{D}(s_2')(\alpha_2', [\,K(k_2') \mid \alpha \mapsto \mathcal{D}(s_1)\,])(\sigma_2') \cup$$
$$\mathcal{D}(s_2')(\alpha_2', [\,K(k_2') \mid \alpha \mapsto \mathcal{D}(s_2)\,])(\sigma_2')) \quad ^{(5.5.5')}$$

Therefore, by taking $max\{\ ^{(5.5.4')}\ ,\ ^{(5.5.5')}\ \}$ we obtain the desired result $^{(5.5.2)}$.
□

The next two lemmas are presented without proofs (which are simple enough and can be used by the reader as proof exercises).

LEMMA 5.6 *For all $k \in KRes$*

(a) $id(k) = id(K(k))$

(b) $\omega_A(k) = \Omega_A(K(k))$

(c) $\omega_S(k) = \Omega_S(K(k))$

(d) $\omega(k) = \Omega(K(k))$

(e) $terminates(k) = Terminates(K(k))$

(f) $blocks(k) = Blocks(K(k))$

(g) $K(k) \setminus \pi = K(k \setminus \pi)$, *for any* $\pi \in \Pi$

(h) $[\, k \mid \alpha \mapsto \theta \,] \setminus \{\alpha'\} = [\, k \setminus \{\alpha'\} \mid \alpha \mapsto \theta \,]$, *for* $\alpha, \alpha' \in Id, \theta \in Comp$, *with* $\alpha' \neq \alpha$

(i) $Sync(\varsigma, K(k)) = sync(\varsigma, k)$, *for any* $\varsigma \in Sched$

(j) $(\, \sigma \mid \varsigma \Rightarrow K(k) \,) = (\, \sigma \mid \varsigma \rightarrow k \,)$, *for any* $\sigma \in \Sigma, \varsigma \in Sched$.

LEMMA 5.7 *Let $k_1, k_2 \in KRes$ be such that $k_1 \cong k_2$. In (a),(b) and (c) $\mu : id(k_1) \rightarrow id(k_2)$ is a bijection that satisfies the properties given in Definition 5.2(a). We have:*

(a) $\omega_A(k_2) = \{(\mu(\alpha)) \mid (\alpha) \in \omega_A(k_1)\}$

(b) $\omega_S(k_2) = \{(\mu(\alpha), \mu(\alpha_1), \cdots, \mu(\alpha_n)) \mid (\alpha, \alpha_1, \cdots, \alpha_n) \in \omega_S(k_1), n \geq 1\}$

(c) $\omega(k_2) = \{(\mu(\alpha_1), \cdots, \mu(\alpha_m)) \mid (\alpha_1, \cdots, \alpha_m) \in \omega(k_1), m \geq 1\}$

(d) $terminates(k_1) = terminates(k_2)$

(e) $blocks(k_1) = blocks(k_2)$

Also, if $\theta \in Comp, (\alpha_1, k_1), (\alpha_2, k_2) \in CRes$, $(\alpha_1, k_1) \cong (\alpha_2, k_2)$ then

(f) $[\, k_1 \mid \alpha_1 \mapsto \theta \,] \cong [\, k_2 \mid \alpha_2 \mapsto \theta \,]$

(g) $(\alpha_1 \cdot 1, [\, k_1 \mid \alpha_1 \mapsto \theta \,]) \cong (\alpha_2 \cdot 1, [\, k_2 \mid \alpha_2 \mapsto \theta \,])$

(h) $(\alpha_1 \cdot 1, [\, k_1 \mid \alpha_1 \cdot 2 \mapsto \theta \,]) \cong (\alpha_2 \cdot 1, [\, k_2 \mid \alpha_2 \cdot 2 \mapsto \theta \,])$

(i) $(\alpha_1 \cdot 2, [\, k_1 \mid \alpha_1 \cdot 1 \mapsto \theta \,]) \cong (\alpha_2 \cdot 2, [\, k_2 \mid \alpha_2 \cdot 1 \mapsto \theta \,])$

If $k_1, k_2 \in KRes$ are such that $k_1 \cong k_2$, $\sigma \in \Sigma$, $\varsigma \in \omega_S(k_1)$ and $\mu : id(k_1) \rightarrow id(k_2)$ is a bijection that satisfies the properties given in Definition 5.2(a) then

(j) $(\, \sigma \mid \varsigma \rightarrow k_1 \,) = (\, \sigma \mid \mu(\varsigma) \rightarrow k_2 \,)$

The following lemma shows that parallel composition is commutative and associative and that sequential composition is associative with respect to all isomorphic continuations that contain only denotations of program statements. In Theorem 5.11 we show that these properties are also preserved in arbitrary syntactic contexts (a notion to be introduced below).

LEMMA 5.8 *If $s_1, s_2, s_3 \in Stat, (\alpha, k), (\overline{\alpha}, \overline{k}) \in CRes$, $(\alpha, k) \cong (\overline{\alpha}, \overline{k})$ and $\sigma \in \Sigma$ then:*

(a) $\mathcal{D}(s_1 \parallel s_2)(\alpha, K(k))(\sigma) = \mathcal{D}(s_2 \parallel s_1)(\overline{\alpha}, K(\overline{k}))(\sigma)$

(b) $\mathcal{D}(s_1 \parallel (s_2 \parallel s_3))(\alpha, K(k))(\sigma) = \mathcal{D}((s_1 \parallel s_2) \parallel s_3)(\overline{\alpha}, K(\overline{k}))(\sigma)$

(c) $\mathcal{D}(s_1; (s_2; s_3))(\alpha, K(k))(\sigma) = \mathcal{D}((s_1; s_2); s_3)(\overline{\alpha}, K(\overline{k}))(\sigma)$

(d) $\mathcal{D}(s_1; (s_2 + s_3))(\alpha, K(k))(\sigma) = \mathcal{D}(s_1; s_2 + s_1; s_3)(\overline{\alpha}, K(\overline{k}))(\sigma)$[8]

Proof:

(a) $d(\mathcal{D}(s_1 \parallel s_2)(\alpha, K(k))(\sigma), \mathcal{D}(s_2 \parallel s_1)(\overline{\alpha}, K(\overline{k}))(\sigma))$

$$
\begin{aligned}
&= d(\mathcal{D}(s_1)(\alpha \cdot 1, [\, K(k) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma) \cup \\
&\qquad \mathcal{D}(s_2)(\alpha \cdot 2, [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_1)\,])(\sigma), \\
&\qquad \mathcal{D}(s_2)(\overline{\alpha} \cdot 1, [\, K(\overline{k}) \mid \overline{\alpha} \cdot 2 \mapsto \mathcal{D}(s_1)\,])(\sigma) \cup \\
&\qquad \mathcal{D}(s_1)(\overline{\alpha} \cdot 2, [\, K(\overline{k}) \mid \overline{\alpha} \cdot 1 \mapsto \mathcal{D}(s_2)\,])(\sigma)) && \text{[Lemma 5.1]} \\
&= d(\mathcal{D}(s_1)(\alpha \cdot 1, K[\, k \mid \alpha \cdot 2 \mapsto s_2\,])(\sigma)\ ^{(5.8.1)} \cup \\
&\qquad \mathcal{D}(s_2)(\alpha \cdot 2, K[\, k \mid \alpha \cdot 1 \mapsto s_1\,])(\sigma)\ ^{(5.8.2)}, \\
&\qquad \mathcal{D}(s_2)(\overline{\alpha} \cdot 1, K[\, \overline{k} \mid \overline{\alpha} \cdot 2 \mapsto s_1\,])(\sigma)\ ^{(5.8.3)} \cup \\
&\qquad \mathcal{D}(s_1)(\overline{\alpha} \cdot 2, K[\, \overline{k} \mid \overline{\alpha} \cdot 1 \mapsto s_2\,])(\sigma)\ ^{(5.8.4)}) \quad \text{['}\cup\text{' is nonexpansive]} \\
&\leq max\{d(\ ^{(5.8.1)}\ ,\ ^{(5.8.4)}\ ), d(\ ^{(5.8.2)}\ ,\ ^{(5.8.3)}\ )\}
\end{aligned}
$$

It is easy to check that $(\alpha, k) \cong (\overline{\alpha}, \overline{k}) \Rightarrow$

$$
\begin{aligned}
&(\alpha \cdot 1, [\, k \mid \alpha \cdot 2 \mapsto s_2\,]) \cong (\overline{\alpha} \cdot 2, [\, \overline{k} \mid \overline{\alpha} \cdot 1 \mapsto s_2\,]) \text{ and} \\
&(\alpha \cdot 2, [\, k \mid \alpha \cdot 1 \mapsto s_1\,]) \cong (\overline{\alpha} \cdot 1, [\, \overline{k} \mid \overline{\alpha} \cdot 2 \mapsto s_1\,]).
\end{aligned}
$$

For example, we can take a bijection that maps $\alpha \cdot 1$ to $\overline{\alpha} \cdot 2$ and $\alpha \cdot 2$ to $\overline{\alpha} \cdot 1$. Therefore, by Corollary 1(a) $max\{d(\ ^{(5.8.1)}\ ,\ ^{(5.8.4)}\ ), d(\ ^{(5.8.2)}\ ,\ ^{(5.8.3)}\ )\} = 0$, which implies $\mathcal{D}(s_1 \parallel s_2)(\alpha, K(k))(\sigma) = \mathcal{D}(s_2 \parallel s_1)(\overline{\alpha}, K(\overline{k}))(\sigma)$.

[8]Recall that ; binds stronger than +, therefore $s_1; s_2 + s_1; s_3$ means $(s_1; s_2) + (s_1; s_3)$.

(b) Notice that it suffices to prove that $\forall(\alpha, k) \in CRes$:

$$\mathcal{D}(s_1\|(s_2\|s_3))(\alpha, K(k))(\sigma) = \mathcal{D}((s_1\|s_2)\|s_3)(\alpha, K(k))(\sigma).$$

because, by Corollary 1(a): $\mathcal{D}((s_1\|s_2)\|s_3)(\alpha, K(k))(\sigma) = \mathcal{D}((s_1\|s_2)\|s_3)(\overline{\alpha}, K(\overline{k}))(\sigma)$. We obtain the desired result by showing that:

$$d(\mathcal{D}(s_1\|(s_2\|s_3))(\alpha, K(k))(\sigma)\ ^{(5.8.5)}, \mathcal{D}((s_1\|s_2)\|s_3)(\alpha, K(k))(\sigma)\ ^{(5.8.6)}) = 0.$$

We expand $^{(5.8.5)}$ and $^{(5.8.6)}$ as follows

$$\begin{aligned}
^{(5.8.5)} &= \mathcal{D}(s_1)(\alpha\cdot 1, [\,K(k) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_2 \parallel s_3)\,])(\sigma) \cup \\
&\quad \mathcal{D}(s_2\|s_3)(\alpha\cdot 2, [\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1)\,])(\sigma) \\
&= \mathcal{D}(s_1)(\alpha\cdot 1, [\,K(k) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_2 \parallel s_3)\,])(\sigma)\ ^{(5.8.7)} \cup \\
&\quad \mathcal{D}(s_2)(\alpha\cdot 2\cdot 1, [\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha\cdot 2\cdot 2 \mapsto \mathcal{D}(s_3)\,])(\sigma)\ ^{(5.8.8)} \cup \\
&\quad \mathcal{D}(s_3)(\alpha\cdot 2\cdot 2, [\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha\cdot 2\cdot 1 \mapsto \mathcal{D}(s_2)\,])(\sigma)\ ^{(5.8.9)} \\
^{(5.8.6)} &= \mathcal{D}(s_1 \parallel s_2)(\alpha\cdot 1, [\,K(k) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_3)\,])(\sigma) \cup \\
&\quad \mathcal{D}(s_3)(\alpha\cdot 2, [\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma) \\
&= \mathcal{D}(s_1)(\alpha\cdot 1\cdot 1, [\,K(k) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_3) \mid \alpha\cdot 1\cdot 2 \mapsto \mathcal{D}(s_2)\,])(\sigma)\ ^{(5.8.10)} \cup \\
&\quad \mathcal{D}(s_2)(\alpha\cdot 1\cdot 2, [\,K(k) \mid \alpha\cdot 2 \mapsto \mathcal{D}(s_3) \mid \alpha\cdot 1\cdot 1 \mapsto \mathcal{D}(s_1)\,])(\sigma)\ ^{(5.8.11)} \cup \\
&\quad \mathcal{D}(s_3)(\alpha\cdot 2, [\,K(k) \mid \alpha\cdot 1 \mapsto \mathcal{D}(s_1 \parallel s_2)\,])(\sigma)\ ^{(5.8.12)}
\end{aligned}$$

We prove that $^{(5.8.7)} = {}^{(5.8.10)}$, $^{(5.8.8)} = {}^{(5.8.11)}$ and $^{(5.8.9)} = {}^{(5.8.12)}$. $^{(5.8.8)} = {}^{(5.8.11)}$ follows easily from Corollary 1(a). Indeed

$$\begin{aligned}
^{(5.8.8)} &= [\text{Lemma 5.1}]\ \mathcal{D}(s_2)(\alpha\cdot 2\cdot 1, K[\,k \mid \alpha\cdot 1 \mapsto s_1 \mid \alpha\cdot 2\cdot 2 \mapsto s_3\,])(\sigma) \\
^{(5.8.11)} &= [\text{Lemma 5.1}]\ \mathcal{D}(s_2)(\alpha\cdot 1\cdot 2, K[\,k \mid \alpha\cdot 2 \mapsto s_3 \mid \alpha\cdot 1\cdot 1 \mapsto s_1\,])(\sigma)
\end{aligned}$$

and it is easy to check that

$$\begin{aligned}
&(\alpha\cdot 2\cdot 1, [\,k \mid \alpha\cdot 1 \mapsto s_1 \mid \alpha\cdot 2\cdot 2 \mapsto s_3\,]) \cong \\
&(\alpha\cdot 1\cdot 2, [\,k \mid \alpha\cdot 2 \mapsto s_3 \mid \alpha\cdot 1\cdot 1 \mapsto s_1\,])
\end{aligned}$$

This isomorphism can be established by considering the following bijection:

$$\begin{aligned}
&\mu : \{\alpha\cdot 2\cdot 1, \alpha\cdot 1, \alpha\cdot 2\cdot 2\} \cup id(k) \to \{\alpha\cdot 1\cdot 2, \alpha\cdot 2, \alpha\cdot 1\cdot 1\} \cup id(k)\ [9] \\
&\mu(\alpha\cdot 2\cdot 1) = \alpha\cdot 1\cdot 2 \\
&\mu(\alpha\cdot 1) = \alpha\cdot 1\cdot 1 \\
&\mu(\alpha\cdot 2\cdot 2) = \alpha\cdot 2
\end{aligned}$$

[9] $(\alpha, k) \in CRes \Rightarrow \alpha\cdot 2\cdot 1, \alpha\cdot 1, \alpha\cdot 2\cdot 2, \alpha\cdot 1\cdot 2, \alpha\cdot 2, \alpha\cdot 1\cdot 1 \notin id(k)$.

$\mu(\alpha') = \alpha'$, for $\alpha' \in id(k)$

In order to approach the relationships $^{(5.8.7)} = {}^{(5.8.10)}$ and $^{(5.8.9)} = {}^{(5.8.12)}$, we notice that:

$$(\alpha \cdot 1 \cdot 1, [\, k \mid \alpha \cdot 2 \mapsto s_3 \mid \alpha \cdot 1 \cdot 2 \mapsto s_2 \,]) \cong$$
$$(\alpha \cdot 1, [\, k \mid \alpha \cdot 2 \cdot 2 \mapsto s_3 \mid \alpha \cdot 2 \cdot 1 \mapsto s_2 \,])$$
$$(\alpha \cdot 2 \cdot 2, [\, k \mid \alpha \cdot 1 \mapsto s_1 \mid \alpha \cdot 2 \cdot 1 \mapsto s_2 \,]) \cong$$
$$(\alpha \cdot 1, [\, k \mid \alpha \cdot 2 \cdot 1 \mapsto s_1 \mid \alpha \cdot 2 \cdot 2 \mapsto s_2 \,])$$
$$(\alpha \cdot 2, [\, k \mid \alpha \cdot 1 \mapsto s_1 \parallel s_2 \,]) \cong (\alpha \cdot 1, [\, k \mid \alpha \cdot 2 \mapsto s_1 \parallel s_2 \,])$$

Therefore, in order to prove $^{(5.8.7)} = {}^{(5.8.10)}$ and $^{(5.8.9)} = {}^{(5.8.12)}$, by Corollary 1(a) and Lemma 5.1 it suffices to show that $\forall s', s_1, s_2 \in Stat, \alpha \in Id, k \in KRes, \sigma \in \Sigma$ such that $(\alpha, k) \in CRes$:

$$\mathcal{D}(s')(\alpha \cdot 1, [\, K(k) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s_1 \parallel s_2) \,])(\sigma) =$$
$$\mathcal{D}(s')(\alpha \cdot 1, [\, K(k) \mid \alpha \cdot 2 \cdot 1 \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \cdot 2 \mapsto \mathcal{D}(s_2) \,])(\sigma)$$

This follows immediately from Lemma 5.3(b).

(c) It suffices to prove the following equality

$$\mathcal{D}(s_1; (s_2; s_3))(\alpha, K(k))(\sigma) = \mathcal{D}((s_1; s_2); s_3)(\alpha, K(k))(\sigma).$$

because, by Corollary 1(a): $\mathcal{D}((s_1; s_2); s_3)(\alpha, K(k))(\sigma) = \mathcal{D}((s_1; s_2); s_3)(\overline{\alpha}, K(\overline{k}))(\sigma)$. We compute as follows:

$$\begin{aligned}
&\mathcal{D}(s_1; (s_2; s_3))(\alpha, K(k))(\sigma) \\
&= \mathcal{D}(s_1)(\alpha \cdot 1, [\, K(k) \mid \alpha \mapsto \mathcal{D}(s_2; s_3) \,])(\sigma)) && \text{[Lemma 5.1]} \\
&= \mathcal{D}(s_1)(\alpha \cdot 1, K[\, k \mid \alpha \mapsto s_2; s_3 \,])(\sigma) \ ^{(5.8.13)}
\end{aligned}$$

It is easy to check that: $(\alpha \cdot 1, [\, k \mid \alpha \mapsto s_2; s_3 \,]) \cong (\alpha \cdot 1 \cdot 1, [\, k \mid \alpha \mapsto s_2; s_3 \,])$. Therefore, by 1(a), $^{(5.8.13)} = \mathcal{D}(s_1)(\alpha \cdot 1 \cdot 1, K[\, k \mid \alpha \mapsto s_2; s_3 \,])(\sigma)$, and

$$\begin{aligned}
&\mathcal{D}(s_1)(\alpha \cdot 1 \cdot 1, K[\, k \mid \alpha \mapsto s_2; s_3 \,])(\sigma) && \text{[Lemma 5.1]} \\
&= \mathcal{D}(s_1)(\alpha \cdot 1 \cdot 1, [\, K(k) \mid \alpha \mapsto \mathcal{D}(s_2; s_3) \,])(\sigma) && \text{[Lemma 5.4(b)]} \\
&= \mathcal{D}(s_1)(\alpha \cdot 1 \cdot 1, [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s_2) \mid \alpha \mapsto \mathcal{D}(s_3) \,])(\sigma) \\
&= \mathcal{D}(s_1; s_2)(\alpha \cdot 1, [\, K(k) \mid \alpha \mapsto \mathcal{D}(s_3) \,])(\sigma) \\
&= \mathcal{D}((s_1; s_2); s_3)(\alpha, K(k))(\sigma)
\end{aligned}$$

(d) By Corollary 1(a) $\mathcal{D}(s_1; s_2 + s_1; s_3)(\alpha, K(k))(\sigma) = \mathcal{D}(s_1; s_2 + s_1; s_3)(\overline{\alpha}, K(\overline{k}))(\sigma)$. Therefore, it suffices to prove:

$$\mathcal{D}(s_1;(s_2+s_3))(\alpha,K(k))(\sigma)=\mathcal{D}(s_1;s_2+s_1;s_3)(\alpha,K(k))(\sigma).$$

We compute as follows:

$$\begin{aligned}
&\mathcal{D}(s_1;(s_2+s_3))(\alpha,K(k))(\sigma)\\
&=\mathcal{D}(s_1)(\alpha\cdot 1,[\,K(k)\mid\alpha\mapsto\mathcal{D}(s_2+s_3)\,])(\sigma)) && \text{[Lemma 5.5(b)]}\\
&=\mathcal{D}(s_1)(\alpha\cdot 1,[\,K(k)\mid\alpha\mapsto\mathcal{D}(s_2)\,])(\sigma)\,\cup\,\mathcal{D}(s_1)(\alpha\cdot 1,[\,K(k)\mid\alpha\mapsto\mathcal{D}(s_3)\,])(\sigma)\\
&=\mathcal{D}(s_1;s2)(\alpha,K(k))(\sigma)\,\cup\,\mathcal{D}(s_1;s_3)(\alpha,K(k))(\sigma)\\
&=\mathcal{D}(s_1;s2+s_1;s_3)(\alpha,K(k))(\sigma) && \square
\end{aligned}$$

Continuations are semantic evaluation contexts [6]. In the sequel we show that CSC continuations can be combined with syntactic contexts to reason upon the behavior of concurrent programs in a compositional manner. In Definition 5.3 we introduce the notion of *syntactic context* for the class of MCC statements.

DEFINITION 5.3 *(Contexts for MCC)*

$$C ::= (\cdot)\mid a\mid x\mid C;C\mid C+C\mid C\|C$$

We denote by $C(s)$ the result of substituting s for $(\cdot)$ in C. Formally, this substitution can be defined inductively: $(\cdot)(s)=s$, $a(s)=a$, $x(s)=x$, and $(C_1\,\mathsf{op}\,C_2)(s)=C_1(s)\,\mathsf{op}\,C_2(s)$ where $\mathsf{op}\in\{;,+,\|\}$.

Lemma 5.9 shows that program properties are preserved in any syntactic context by all CSC continuations containing only computations denotable by program statements. In the proof of Lemma 5.9 we use some results presented later in Lemma 5.10.

LEMMA 5.9 *If $s_1,s_2\in Stat$ such that $\mathcal{D}(s_1)(\alpha,K(k))(\sigma)=\mathcal{D}(s_2)(\alpha,K(k))(\sigma)$ for all $(\alpha,k)\in CRes$ and $\sigma\in\Sigma$, then we have*

$$\mathcal{D}(C(s_1))(\alpha,K(k))(\sigma)=\mathcal{D}(C(s_2))(\alpha,K(k))(\sigma)$$

for all $(\alpha,k)\in CRes,\sigma\in\Sigma$ and all contexts C.

Proof: By structural induction on C. Cases $[C=a]$ and $[C=x]$ are trivial; the case $[C=(\cdot)]$ follows immediately by the inductive assumption. We also treat the following subcases.

Case $[C=C_1+C_2]$

$$\begin{aligned}
&\mathcal{D}((C_1+C_2)(s_1))(\alpha,K(k))(\sigma)\\
&=\mathcal{D}(C_1(s_1)+C_2(s_1))(\alpha,K(k))(\sigma)\\
&=\mathcal{D}(C_1(s_1))(\alpha,K(k))(\sigma)\cup\mathcal{D}(C_2(s_1))(\alpha,K(k))(\sigma) && \text{[Ind. hyp.]}\\
&=\mathcal{D}(C_1(s_2))(\alpha,K(k))(\sigma)\cup\mathcal{D}(C_2(s_2))(\alpha,K(k))(\sigma)
\end{aligned}$$

$$= \mathcal{D}(C_1(s_2) + C_2(s_2))(\alpha, K(k))(\sigma)$$
$$= \mathcal{D}((C_1 + C_2)(s_2))(\alpha, K(k))(\sigma)$$

Case $[C = C_1 \parallel C_2]$. Let $(\alpha, k) \in CRes, \sigma \in \Sigma$. We have to prove that:

$$\mathcal{D}((C_1 \parallel C_2)(s_1))(\alpha, K(k))(\sigma) = \mathcal{D}((C_1 \parallel C_2)(s_2))(\alpha, K(k))(\sigma)$$

We compute as follows:

$$\begin{aligned}
&\mathcal{D}((C_1 \parallel C_2)(s_1))(\alpha, K(k))(\sigma) \\
&= \mathcal{D}(C_1(s_1) \parallel C_2(s_1))(\alpha, K(k))(\sigma) \\
&= \mathcal{D}(C_1(s_1))(\alpha \cdot 1, [\, K(k) \mid \alpha \cdot 2 \mapsto \mathcal{D}(C_2(s_1))\,])(\sigma)\ ^{(5.9.1)} \cup \\
&\quad \mathcal{D}(C_2(s_1))(\alpha \cdot 2, [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(C_1(s_1))\,])(\sigma)\ ^{(5.9.2)}
\end{aligned}$$

We handle $^{(5.9.1)}$ first.

$$\begin{aligned}
&^{(5.9.1)} = \mathcal{D}(C_1(s_1))(\alpha \cdot 1, [\, K(k) \mid \alpha \cdot 2 \mapsto \mathcal{D}(C_2(s_1))\,])(\sigma) && \text{[Lemma 5.1]} \\
&= \mathcal{D}(C_1(s_1))(\alpha \cdot 1, K[\, k \mid \alpha \cdot 2 \mapsto (C_2(s_1))\,])(\sigma) && \text{[Ind. hyp.]} \\
&= \mathcal{D}(C_1(s_2))(\alpha \cdot 1, K[\, k \mid \alpha \cdot 2 \mapsto (C_2(s_1))\,])(\sigma) && \text{[Lemma 5.1]} \\
&= \mathcal{D}(C_1(s_2))(\alpha \cdot 1, [\, K(k) \mid \alpha \cdot 2 \mapsto \mathcal{D}(C_2(s_1))\,])(\sigma)
\end{aligned}$$

$(\alpha, k) \in CRes$ implies $(\alpha \cdot 1, k) \in CRes$. Hence, by the induction hypothesis $\mathcal{D}(C_2(s_1))(\alpha \cdot 1, K(k))(\sigma) = \mathcal{D}(C_2(s_2))(\alpha \cdot 1, K(k))(\sigma)$. By Lemma 5.10(b)

$$\begin{aligned}
&\mathcal{D}(C_1(s_2))(\alpha \cdot 1, [\, K(k) \mid \alpha \cdot 2 \mapsto \mathcal{D}(C_2(s_1))\,])(\sigma) \\
&= \mathcal{D}(C_1(s_2))(\alpha \cdot 1, [\, K(k) \mid \alpha \cdot 2 \mapsto \mathcal{D}(C_2(s_2))\,])(\sigma)
\end{aligned}$$

Similarly

$$\begin{aligned}
&^{(5.9.2)} = \mathcal{D}(C_2(s_1))(\alpha \cdot 2, [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(C_1(s_1))\,])(\sigma) \\
&= \mathcal{D}(C_2(s_2))(\alpha \cdot 2, [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(C_1(s_2))\,])(\sigma)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathcal{D}((C_1 \parallel C_2)(s_1))(\alpha, K(k))(\sigma) = {}^{(5.9.1)} \cup {}^{(5.9.2)} \\
&= \mathcal{D}(C_1(s_2))(\alpha \cdot 1, [\, K(k) \mid \alpha \cdot 2 \mapsto \mathcal{D}(C_2(s_2))\,])(\sigma) \cup \\
&\quad \mathcal{D}(C_2(s_2))(\alpha \cdot 2, [\, K(k) \mid \alpha \cdot 1 \mapsto \mathcal{D}(C_1(s_2))\,])(\sigma) \\
&= \mathcal{D}(C_1(s_2) \parallel C_2(s_2))(\alpha, K(k))(\sigma) \\
&= \mathcal{D}((C_1 \parallel C_2)(s_2))(\alpha, K(k))(\sigma)
\end{aligned}$$

□

LEMMA 5.10 *If $s_1, s_2 \in Stat$ are such that for all $(\alpha, k) \in CRes, \sigma \in \Sigma$:*

$$\mathcal{D}(s_1)(\alpha, K(k))(\sigma) = \mathcal{D}(s_2)(\alpha, K(k))(\sigma)$$

then

(a) for all $k \in KRes, \overline{\alpha} \in Id, \sigma \in \Sigma$

$$kc([\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma) = kc([\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma)$$

(b) for all $s \in Stat, (\alpha, k) \in CRes, \sigma \in \Sigma, \overline{\alpha} \in Id, \neg(\overline{\alpha} \geq \alpha)$*:*

$$\mathcal{D}(s)(\alpha, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma) = \mathcal{D}(s)(\alpha, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma)$$

Proof: For (a) we show that $\forall k \in KRes, \sigma \in \Sigma, \overline{\alpha} \in Id, \quad \exists s' \in Stat, \alpha' \in Id, k' \in KRes, \neg(\overline{\alpha} \geq \alpha')$ such that:

$$\begin{aligned}
&\epsilon^C_{kc}(k, \overline{\alpha}, s_1, s_2, \sigma) \stackrel{not.}{=} \\
&d(kc([\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma), kc([\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma)) \\
&\leq d(\mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma'), \mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma')) \\
&\stackrel{not.}{=} \epsilon^C_{\mathcal{D}}(s', \alpha', k', \overline{\alpha}, s_1, s_2, \sigma') \ {}^{(5.10.1)}
\end{aligned}$$

For (b) we show that $\forall s \in Stat, \alpha \in Id, k \in KRes, \overline{\alpha} \in Id, \sigma \in \Sigma, \neg(\overline{\alpha} \geq \alpha), \quad \exists s' \in Stat, \alpha' \in Id, k' \in KRes, \sigma' \in \Sigma, \neg(\overline{\alpha} \geq \alpha')$ such that:[10]

$$\begin{aligned}
&\epsilon^C_{\mathcal{D}}(s, \alpha, k, \overline{\alpha}, s_1, s_2, \sigma) \\
&= d(\mathcal{D}(s)(\alpha, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma), \mathcal{D}(s)(\alpha, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma)) \\
&\leq \tfrac{1}{2} \cdot \ d(\mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma'), \mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma')) \\
&= \tfrac{1}{2} \cdot \ \epsilon^C_{\mathcal{D}}(s', \alpha', k', \overline{\alpha}, s_1, s_2, \sigma') \ {}^{(5.10.2)}
\end{aligned}$$

If we put $(w \in)W = Stat \times Id \times KRes \times Id \times Stat \times Stat \times \Sigma$ we infer that

$$sup_{w \in W : \neg(\overline{\alpha} \geq \alpha)} \epsilon^C_{\mathcal{D}}(w) \leq \tfrac{1}{2} \cdot \ sup_{w' \in W : \neg(\overline{\alpha} \geq \alpha')} \epsilon^C_{\mathcal{D}}(w')$$

where $w = (s, \alpha, k, \overline{\alpha}, s_1, s_2, \sigma), w' = (s', \alpha', k', \overline{\alpha}, s_1, s_2, \sigma')$. But

$$sup_{w \in W : \neg(\overline{\alpha} \geq \alpha)} \epsilon^C_{\mathcal{D}}(w) = sup_{w' \in W : \neg(\overline{\alpha} \geq \alpha')} \epsilon^C_{\mathcal{D}}(w')$$

Therefore we infer $sup_{w \in W : \neg(\overline{\alpha} \geq \alpha)} \epsilon^C_{\mathcal{D}}(w) = 0$, which implies Lemma 5.10(b). Next, by using this result and $^{(5.10.1)}$ we obtain immediately Lemma 5.10(a). In the sequel we prove $^{(5.10.1)}$ and $^{(5.10.2)}$.

First, we prove $^{(5.10.1)}$ (i.e. $\epsilon^C_{kc}(k, \overline{\alpha}, s_1, s_2, \sigma) \leq \epsilon^C_{\mathcal{D}}(s', \alpha', k', \overline{\alpha}, s_1, s_2, \sigma')$. Let

$$\kappa_1 = [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,] \qquad (= K[\, k \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,], \text{ by Lemma 5.1})$$

[10] $\neg(\overline{\alpha} \geq \alpha)$ is the invariant property that is preserved by the computation steps.

$\kappa_2 = [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,] \qquad (= K[\, k \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,]$, by Lemma 5.1)

We compute as follows:

$$\epsilon^C_{kc}(k, \overline{\alpha}, s_1, s_2, \sigma) = d(kc(\kappa_1)(\sigma), kc(\kappa_2)(\sigma))$$

Obviously, $\neg Terminates(\kappa_1)$ and $\neg Terminates(\kappa_2)$. Notice that:

$$\Omega(\kappa_1) = \Omega(\kappa_2)$$

$$= \ \textsf{if}\ (\overline{\alpha} \in (max(id(k) \cup \{\overline{\alpha}\})))$$

$$\textsf{then}\ \omega(k) \cup \{\overline{\alpha}\} \setminus$$

$$\{\{\alpha_1, \cdots, \alpha_m\} \mid \{\alpha_1, \cdots, \alpha_m\} \in \omega(k), \exists 1 \le i \le m : \overline{\alpha} \ge \alpha_i\}$$

$$\textsf{else}\ \omega(k)$$

We prove $^{(5.10.1)}$ by induction on $|I(k)|$ where:[11]

$$I(k) = \{\alpha \mid \alpha \in id(k), k(\alpha) \in J^{\diamondsuit}\}$$

We begin with the case $|I(k)| = 0$. In this case $\Omega_S(\kappa_1) = \Omega_S(\kappa_2) = \emptyset$, and $\Omega(\kappa_1) = \Omega_A(\kappa_1) = \Omega_A(\kappa_2) = \Omega(\kappa_2) \neq \emptyset$, which implies $\neg Blocks(\kappa_1)$ and $\neg Blocks(\kappa_2)$. Two subcases. If $(\overline{\alpha} \notin (max(id(k) \cup \{\overline{\alpha}\})))$ then

$$d(kc(\kappa_1)(\sigma), kc(\kappa_2)(\sigma))$$

$$= d(\textstyle\bigcup_{\{\alpha\} \in \Omega_A(\kappa_1)} \kappa_1(\alpha)(\alpha, \kappa_1 \setminus \{\alpha\})(\sigma),$$

$$\textstyle\bigcup_{\{\alpha\} \in \Omega_A(\kappa_2)} \kappa_2(\alpha)(\alpha, \kappa_2 \setminus \{\alpha\})(\sigma))$$

['$\cup$' is nonexpansive]

$$\le max\{d(\kappa_1(\alpha)(\alpha, \kappa_1 \setminus \{\alpha\})(\sigma), \kappa_2(\alpha)(\alpha, \kappa_2 \setminus \{\alpha\})(\sigma)) \mid \{\alpha\} \in \omega(k)\} \ ^{(5.10.1.1)}$$

Let $s_\alpha = k(\alpha), \forall \{\alpha\} \in \omega(k)$. As $\overline{\alpha} \notin (max(id(k) \cup \{\overline{\alpha}\}))$ it follows that $\{\overline{\alpha}\} \notin \omega(k)$. Also, notice that $\kappa_1 \setminus \{\alpha\} = [\, K(k \setminus \{\alpha\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \,]$, $\kappa_2 \setminus \{\alpha\} = [\, K(k \setminus \{\alpha\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,]$. Therefore

$$^{(5.10.1.1)} = max\{d(\mathcal{D}(s_\alpha)(\alpha, [\, K(k \setminus \{\alpha\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \,])(\sigma),$$

$$\mathcal{D}(s_\alpha)(\alpha, [\, K(k \setminus \{\alpha\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,])(\sigma)) \mid \{\alpha\} \in \omega(k)\}$$

Clearly, this means that $\exists s' \in Stat, \alpha' \in Id, k' \in KRes, \sigma' \in \Sigma$ such that:[12]

$$\epsilon^C_{kc}(k, \overline{\alpha}, s_1, s_2, \sigma) = d(kc(\kappa_1)(\sigma), kc(\kappa_2)(\sigma))$$

[11] Recall that $|\cdot|$ is the cardinal number of '$\cdot$'.

[12] More precisely, for some $\alpha \in Id$ such that $\{\alpha\} \in \omega(k)$, $s' = s_\alpha, \alpha' = \alpha, k' = k \setminus \{\alpha\}, \sigma' = \sigma$.

$$\leq d(\mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma'), \mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma'))$$

$$= \epsilon^C_{\mathcal{D}}(s', \alpha', k', \overline{\alpha}, s_1, s_2, \sigma')$$

Notice that the invariant property $\neg(\overline{\alpha} \geq \alpha')$ is preserved, because $\alpha' = \alpha$ for some $\{\alpha\} \in \omega(k)$, and $\overline{\alpha} \notin (max(id(k) \cup \{\overline{\alpha}\}))$. Therefore (5.10.1) holds in this (sub)case.

Next, we treat the subcase when $\overline{\alpha} \in (max(id(k) \cup \{\overline{\alpha}\}))$. In this subcase:

$$d(kc(\kappa_1)(\sigma), kc(\kappa_2)(\sigma))$$

$$= d(\textstyle\bigcup_{\{\alpha\}\in\omega(k),\neg(\overline{\alpha}\geq\alpha)} \kappa_1(\alpha)(\alpha, \kappa_1 \setminus \{\alpha\})(\sigma) \cup \mathcal{D}(s_1)(\overline{\alpha}, \kappa_1 \setminus \{\overline{\alpha}\})(\sigma),$$

$$\textstyle\bigcup_{\{\alpha\}\in\omega(k),\neg(\overline{\alpha}\geq\alpha)} \kappa_2(\alpha)(\alpha, \kappa_2 \setminus \{\alpha\})(\sigma) \cup \mathcal{D}(s_2)(\overline{\alpha}, \kappa_2 \setminus \{\overline{\alpha}\})(\sigma))$$

['$\cup$' is nonexpansive, Corollary 2]

$$\leq max\{d(\mathcal{D}(s_1)(\overline{\alpha}, K(k))(\sigma), \mathcal{D}(s_2)(\overline{\alpha}, K(k))(\sigma)),$$

$$max\{d(\kappa_1(\alpha)(\alpha, \kappa_1 \setminus \{\alpha\})(\sigma), \kappa_2(\alpha)(\alpha, \kappa_2 \setminus \{\alpha\})(\sigma))$$

$$\mid \{\alpha\} \in \omega(k), \neg(\overline{\alpha} \geq \alpha)\}\} \quad (5.10.1.2)$$

By the assumption of Lemma 5.10, $d(\mathcal{D}(s_1)(\overline{\alpha}, K(k))(\sigma), \mathcal{D}(s_2)(\overline{\alpha}, K(k))(\sigma)) = 0$. So let $s_\alpha = k(\alpha), \forall\{\alpha\} \in \omega(k), \neg(\overline{\alpha} \geq \alpha)$. For any $\{\alpha\} \in \omega(k)$, such that $\neg(\overline{\alpha} \geq \alpha)$, $\kappa_1 \setminus \{\alpha\} = [\, K(k \setminus \{\alpha\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,]$ and $\kappa_2 \setminus \{\alpha\} = [\, K(k \setminus \{\alpha\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,]$. Therefore

$$(5.10.1.2) \;= max\{d(\mathcal{D}(s_\alpha)(\alpha, [\, K(k \setminus \{\alpha\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma),$$

$$\mathcal{D}(s_\alpha)(\alpha, [\, K(k \setminus \{\alpha\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma))$$

$$\mid \{\alpha\} \in \omega(k), \neg(\overline{\alpha} \geq \alpha)\}$$

Clearly, this means that $\exists s' \in Stat, \alpha' \in Id, k' \in KRes, \sigma' \in \Sigma$ (more precisely, for some $\alpha \in Id$ such that $\{\alpha\} \in \omega(k)$ with $\neg(\overline{\alpha} \geq \alpha)$, $s' = s_\alpha, \alpha' = \alpha, k' = k \setminus \{\alpha\}, \sigma' = \sigma$) such that:

$$\epsilon^C_{kc}(k, \overline{\alpha}, s_1, s_2, \sigma)$$

$$= d(kc(\kappa_1)(\sigma), kc(\kappa_2)(\sigma))$$

$$= d(kc([\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma), kc([\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma))$$

$$\leq d(\mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma'), \mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma'))$$

$$= \epsilon^C_{\mathcal{D}}(s', \alpha', k', \overline{\alpha}, s_1, s_2, \sigma')$$

Again, notice that $\neg(\overline{\alpha} \geq \alpha')$ (because $\alpha' = \alpha$) therefore the invariant is preserved by the computation and thus $^{(5.10.1)}$ holds in this (sub)case also.

The induction step follows. So assume that $|I(k)| > 0$, in which case $\Omega(\kappa_1) = \Omega_A(\kappa_1) \cup \Omega_S(\kappa_1)$ and $\Omega(\kappa_2) = \Omega_A(\kappa_2) \cup \Omega_S(\kappa_2)$. In this case at least one of $\Omega_A(\kappa_1)$ or $\Omega_S(\kappa_1)$ (respectively $\Omega_A(\kappa_2)$ or $\Omega_S(\kappa_2)$) is nonempty. Also, it is easy to see that $Blocks(\kappa_1) = Blocks(\kappa_2)$, and obviously $\neg Terminates(\kappa_1)$ and $\neg Terminates(\kappa_2)$. The case when $Blocks(\kappa_1) = Blocks(\kappa_2) = True$ is clear, because $\epsilon^C_{kc}(k, \overline{\alpha}, s_1, s_2, \sigma) = 0$. When $\neg Blocks(\kappa_1)$ and $\neg Blocks(\kappa_2)$ we compute as follows. We consider two subcases. First, we treat the subcase when $\overline{\alpha} \notin (max(id(k) \cup \{\overline{\alpha}\}))$, and recall that in this case $\Omega(\kappa_1) = \Omega(\kappa_2) = \omega(k)$, and also $\Omega_A(\kappa_1) = \Omega_A(\kappa_2) = \omega_A(k)$ and $\Omega_S(\kappa_1) = \Omega_S(\kappa_2) = \omega_S(k)$. We have:

$$
\begin{aligned}
& d(kc(\kappa_1)(\sigma), kc(\kappa_2)(\sigma)) \\
& = d(\textstyle\bigcup_{\{\alpha\} \in \omega_A(k)} \kappa_1(\alpha)(\alpha, \kappa_1 \setminus \{\alpha\})(\sigma) \cup \\
& \qquad \textstyle\bigcup_{\varsigma \in \omega_S(k)} (\, \sigma \mid \varsigma \Rightarrow \kappa_1 \,) \cdot kc(\kappa_1 \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow \kappa_1 \,), \\
& \qquad \textstyle\bigcup_{\{\alpha\} \in \omega_A(k)} \kappa_2(\alpha)(\alpha, \kappa_2 \setminus \{\alpha\})(\sigma) \cup \\
& \qquad \textstyle\bigcup_{\varsigma \in \omega_S(k)} (\, \sigma \mid \varsigma \Rightarrow \kappa_2 \,) \cdot kc(\kappa_2 \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow \kappa_2 \,)) \\
& [\text{'}\cup\text{' is nonexpansive}] \\
& \leq max\{max\{d(\kappa_1(\alpha)(\alpha, \kappa_1 \setminus \{\alpha\})(\sigma), \kappa_2(\alpha)(\alpha, \kappa_2 \setminus \{\alpha\})(\sigma)) \\
& \qquad\qquad \mid \{\alpha\} \in \omega_A(k)\} \; ^{(5.10.1.3)} \;, \\
& \qquad max\{d((\, \sigma \mid \varsigma \Rightarrow \kappa_1 \,) \cdot kc(\kappa_1 \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow \kappa_1 \,), \\
& \qquad\qquad (\, \sigma \mid \varsigma \Rightarrow \kappa_2 \,) \cdot kc(\kappa_2 \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow \kappa_2 \,)) \\
& \qquad\qquad \mid \varsigma \in \omega_S(k)\} \; ^{(5.10.1.4)} \; \}
\end{aligned}
$$

If $^{(5.10.1.3)} \geq {}^{(5.10.1.4)}$ then we proceed as in the basic case (i.e. $|I(k)| = 0$) and we obtain the desired result. Next, assume that $^{(5.10.1.3)} < {}^{(5.10.1.4)}$. We compute as follows. First, we notice that for any $\varsigma = \{\alpha, \alpha_1, \cdots, \alpha_n\} \in \omega_S(k)$ we have $\sigma_{\varsigma k} = (\, \sigma \mid \varsigma \Rightarrow \kappa_1 \,) = (\, \sigma \mid \varsigma \Rightarrow \kappa_2 \,)$, where $k(\alpha) = c_1?v_1 \,\&\, \cdots \,\&\, c_n?v_n = \kappa_1(\alpha) = \kappa_2(\alpha)$, $k(\alpha_i) = \kappa_1(\alpha_i) = \kappa_2(\alpha_i) = c_i!\xi_i, 1 \leq i \leq n$. Hence:

$$
\begin{aligned}
^{(5.10.1.4)} \; & = max\{d(\sigma_{\varsigma k} \cdot kc(\kappa_1 \setminus \widehat{\varsigma})(\sigma_{\varsigma k}), \sigma_{\varsigma k} \cdot kc(\kappa_2 \setminus \widehat{\varsigma})(\sigma_{\varsigma k})) \\
& \qquad \mid \varsigma = \{\alpha, \alpha_1, \cdots, \alpha_n\} \in \omega_S(k), \\
& \qquad k(\alpha) = c_1?v_1 \,\&\, \cdots \,\&\, c_n?v_n, \\
& \qquad k(\alpha_1) = c_1!\xi_1, \cdots, k(\alpha_n) = c_n!\xi_n,
\end{aligned}
$$

$$\sigma_{\varsigma k} = [\,\sigma \mid v_1 \mapsto \xi_1(\sigma) \mid \cdots \mid v_n \mapsto \xi_n(\sigma)\,]\}$$

$$[\overline{\alpha} \notin max(id(k) \cup \{\overline{\alpha}\}), \varsigma \in \omega_S(k) \Rightarrow$$

$$\kappa_1 \setminus \widehat{\varsigma} = [\,K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,] \setminus \widehat{\varsigma} = [\,K(k \setminus \widehat{\varsigma}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,] \text{ and}$$

$$\kappa_2 \setminus \widehat{\varsigma} = [\,K(k \setminus \widehat{\varsigma}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,]\,]$$

$$= \tfrac{1}{2} \cdot max\{d(kc[\,K(k \setminus \{\alpha, \alpha_1, \cdots, \alpha_n\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,](\sigma_{\varsigma k}),$$

$$kc[\,K(k \setminus \{\alpha, \alpha_1, \cdots, \alpha_n\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,](\sigma_{\varsigma k}))$$

$$\mid \{\alpha, \alpha_1, \cdots, \alpha_n\} \in \omega_S(k),$$

$$k(\alpha) = c_1?v_1 \,\&\, \cdots \,\&\, c_n?v_n,$$

$$k(\alpha_1) = c_1!\xi_1, \cdots, k(\alpha_n) = c_n!\xi_n,$$

$$\sigma_{\varsigma k} = [\,\sigma \mid v_1 \mapsto \xi_1(\sigma) \mid \cdots \mid v_n \mapsto \xi_n(\sigma)\,]\}$$

Now notice that $\forall \{\alpha, \alpha_1, \cdots, \alpha_n\} \in \omega_S(k) : |I(k \setminus \{\alpha, \alpha_1, \cdots, \alpha_n\})| < |I(k)|$. Therefore, by the induction hypothesis $\forall \{\alpha, \alpha_1, \cdots, \alpha_n\} \in \omega_S(k)$, $\exists \overline{s}' \in Stat, \overline{\alpha}' \in Id, \overline{k}' \in KRes, \overline{\sigma}' \in \Sigma$, $\neg(\overline{\alpha} \geq \overline{\alpha}')$ such that:

$$d(kc[\,K(k \setminus \{\alpha, \alpha_1, \cdots, \alpha_n\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,](\sigma_{\varsigma k}),$$

$$kc[\,K(k \setminus \{\alpha, \alpha_1, \cdots, \alpha_n\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,](\sigma_{\varsigma k}))$$

$$\leq d(\mathcal{D}(\overline{s}')(\overline{\alpha}', [\,K(\overline{k}') \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\overline{\sigma}'), \mathcal{D}(\overline{s}')(\overline{\alpha}', [\,K(\overline{k}') \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\overline{\sigma}'))$$

It follows immediately that $\exists s' \in Stat, \alpha' \in Id, k' \in KRes, \sigma' \in \Sigma$, $\neg(\overline{\alpha} \geq \alpha')$, such that:

$$\epsilon^C_{kc}(k, \overline{\alpha}, s_1, s_2, \sigma)$$

$$= d(kc(\kappa_1)(\sigma), kc(\kappa_2)(\sigma))$$

$$\leq^{(5.10.1.4)}$$

$$\leq d(\mathcal{D}(s')(\alpha', [\,K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma'), \mathcal{D}(s')(\alpha', [\,K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma'))$$

$$= \epsilon^C_{\mathcal{D}}(s', \alpha', k', \overline{\alpha}, s_1, s_2, \sigma')$$

Next, we consider the subcase when $\overline{\alpha} \in (max(id(k) \cup \{\overline{\alpha}\}))$. Let $P_{sched} : (Id \times Sched) \to Bool$, $P_{sched}(\overline{\alpha}, \{\alpha_1, \cdots, \alpha_m\}) = \forall 1 \leq i \leq m : \neg(\overline{\alpha} \geq \alpha_i)$. In this (sub)case we have:

$$\epsilon^C_{kc}(k, \overline{\alpha}, s_1, s_2, \sigma) = d(kc(\kappa_1)(\sigma), kc(\kappa_2)(\sigma))$$

$$= d(\textstyle\bigcup_{\{\alpha\} \in \omega_A(k), \neg(\overline{\alpha} \geq \alpha)} \kappa_1(\alpha)(\alpha, \kappa_1 \setminus \{\alpha\})(\sigma) \cup$$

$$\mathcal{D}(s_1)(\overline{\alpha}, \kappa_1 \setminus \{\overline{\alpha}\})(\sigma) \cup$$

$$\bigcup_{\varsigma \in \omega_S(k), P_{sched}(\overline{\alpha}, \varsigma)} (\, \sigma \mid \varsigma \Rightarrow \kappa_1 \,) \cdot kc(\kappa_1 \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow \kappa_1 \,),$$

$$\bigcup_{\{\alpha\} \in \omega_A(k), \neg(\overline{\alpha} \geq \alpha)} \kappa_2(\alpha)(\alpha, \kappa_2 \setminus \{\alpha\})(\sigma) \cup$$

$$\mathcal{D}(s_2)(\overline{\alpha}, \kappa_2 \setminus \{\overline{\alpha}\})(\sigma) \cup$$

$$\bigcup_{\varsigma \in \omega_S(k), P_{sched}(\overline{\alpha}, \varsigma)} (\, \sigma \mid \varsigma \Rightarrow \kappa_2 \,) \cdot kc(\kappa_2 \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow \kappa_2 \,))$$

['$\cup$' is nonexpansive, Corollary 2]

$$\leq max\{d(\mathcal{D}(s_1)(\overline{\alpha}, K(k))(\sigma), \mathcal{D}(s_2)(\overline{\alpha}, K(k))(\sigma)),$$

$$max\{d(\kappa_1(\alpha)(\kappa_1 \setminus \{\alpha\})(\sigma), \kappa_2(\alpha)(\kappa_2 \setminus \{\alpha\})(\sigma))$$

$$\mid \{\alpha\} \in \omega_A(k), \neg(\overline{\alpha} \geq \alpha)\} \ ^{(5.10.1.6)}\ ,$$

$$max\{d((\, \sigma \mid \varsigma \Rightarrow \kappa_1 \,) \cdot kc(\kappa_1 \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow \kappa_1 \,),$$

$$(\, \sigma \mid \varsigma \Rightarrow \kappa_2 \,) \cdot kc(\kappa_2 \setminus \widehat{\varsigma})(\, \sigma \mid \varsigma \Rightarrow \kappa_2 \,))$$

$$\mid \varsigma \in \omega_S(k), P_{sched}(\overline{\alpha}, \varsigma)\} \ ^{(5.10.1.7)}\ \}$$

By the assumption of Lemma 5.10, $d(\mathcal{D}(s_1)(\overline{\alpha}, K(k))(\sigma), \mathcal{D}(s_2)(\overline{\alpha}, K(k))(\sigma)) = 0$. Next, if $^{(5.10.1.6)} \geq {}^{(5.10.1.7)}$ then we proceed as in the basic case (i.e. $|I(k)| = 0$) and we obtain the desired result. Finaly, assume that $^{(5.10.1.6)} < {}^{(5.10.1.7)}$. We compute as follows. First, we notice that for any $\varsigma = \{\alpha, \alpha_1, \cdots, \alpha_n\} \in \omega_S(k)$ we have $\sigma_{\varsigma k} = (\, \sigma \mid \varsigma \Rightarrow \kappa_1 \,) = (\, \sigma \mid \varsigma \Rightarrow \kappa_2 \,)$, where $\sigma_{\varsigma k} = [\, \sigma \mid v_1 \mapsto \xi_1(\sigma) \mid \cdots \mid v_n \mapsto \xi_n(\sigma) \,]$, $k(\alpha) = c_1?v_1 \,\&\, \cdots \,\&\, c_n?v_n = \kappa_1(\alpha) = \kappa_2(\alpha)$, $k(\alpha_i) = \kappa_1(\alpha_i) = \kappa_2(\alpha_i) = c_i!\xi_i, 1 \leq i \leq n$. Hence:

$$^{(5.10.1.7)} = max\{d(\sigma_{\varsigma k} \cdot kc(\kappa_1 \setminus \{\alpha, \alpha_1, \cdots, \alpha_n\})(\sigma_{\varsigma k}),$$

$$\sigma_{\varsigma k} \cdot kc(\kappa_2 \setminus \{\alpha, \alpha_1, \cdots, \alpha_n\})(\sigma_{\varsigma k}))$$

$$\mid \{\alpha, \alpha_1, \cdots, \alpha_n\} \in \omega_S(k),$$

$$\neg(\overline{\alpha} \geq \alpha), \forall 1 \leq i \leq n : \neg(\overline{\alpha} \geq \alpha_i),$$

$$k(\alpha) = c_1?v_1 \& \cdots \& c_n?v_n,$$

$$k(\alpha_1) = c_1!\xi_1, \cdots, k(\alpha_n) = c_n!\xi_n,$$

$$\sigma_{\varsigma k} = [\, \sigma \mid v_1 \mapsto \xi_1(\sigma) \mid \cdots \mid v_n \mapsto \xi_n(\sigma) \,]\}$$

$$= \tfrac{1}{2} \cdot max\{d(kc([\, K(k \setminus \{\alpha, \alpha_1, \cdots, \alpha_n\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \,])(\sigma_{\varsigma k}),$$

$$kc([\, K(k \setminus \{\alpha, \alpha_1, \cdots, \alpha_n\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,])(\sigma_{\varsigma k}))$$

$$
\begin{aligned}
&\mid \{\alpha, \alpha_1, \cdots, \alpha_n\} \in \omega_S(k),\\
&\quad \neg(\overline{\alpha} \geq \alpha), \forall 1 \leq i \leq n : \neg(\overline{\alpha} \geq \alpha_i),\\
&\quad k(\alpha) = c_1?v_1 \& \cdots \& c_n?v_n,\\
&\quad k(\alpha_1) = c_1!\xi_1, \cdots, k(\alpha_n) = c_n!\xi_n,\\
&\quad \sigma_{\varsigma k} = [\, \sigma \mid v_1 \mapsto \xi_1(\sigma) \mid \cdots \mid v_n \mapsto \xi_n(\sigma)\,]\}
\end{aligned}
$$

Notice that $\forall \{\alpha, \alpha_1, \cdots, \alpha_n\} \in \omega_S(k) : |I(k \setminus \{\alpha, \alpha_1, \cdots, \alpha_n\})| < |I(k)|$. Therefore, by the induction hypothesis $\forall \{\alpha, \alpha_1, \cdots, \alpha_n\} \in \omega_S(k), \exists \overline{s}' \in Stat, \overline{\alpha}' \in Id, \overline{k}' \in KRes, \overline{\sigma}' \in \Sigma, \neg(\overline{\alpha} \geq \overline{\alpha}')$, such that:

$$
\begin{aligned}
&d(kc([\, K(k \setminus \{\alpha, \alpha_1, \cdots, \alpha_n\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma_{\varsigma k}),\\
&\quad kc([\, K(k \setminus \{\alpha, \alpha_1, \cdots, \alpha_n\}) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma_{\varsigma k}))\\
&\leq d(\mathcal{D}(\overline{s}')(\overline{\alpha}', [\, K(\overline{k}') \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\overline{\sigma}'), \mathcal{D}(\overline{s}')(\overline{\alpha}', [\, K(\overline{k}') \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\overline{\sigma}'))
\end{aligned}
$$

It follows immediately that $\exists s' \in Stat, \alpha' \in Id, k' \in KRes, \sigma' \in \Sigma$, $\neg(\overline{\alpha} \geq \alpha')$, such that:

$$
\begin{aligned}
&(5.10.1.1) \leq (5.10.1.7)\\
&\leq d(\mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma'), \mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma'))\\
&= \epsilon^C_{\mathcal{D}}(s', \alpha', k', \overline{\alpha}, s_1, s_2, \sigma')
\end{aligned}
$$

which completes the proof of (5.10.1).

In the sequel we prove (5.10.2), i.e. we show that $\forall s \in Stat, \alpha \in Id, k \in KRes, \overline{\alpha} \in Id, \sigma \in \Sigma$ with $\neg(\overline{\alpha} \geq \alpha)$, $\exists s' \in Stat, \alpha' \in Id, k' \in KRes, \sigma' \in \Sigma$ with $\neg(\overline{\alpha} \geq \alpha')$ such that:

$$
\epsilon^C_{\mathcal{D}}(s, \alpha, k, \overline{\alpha}, s_1, s_2, \sigma) \leq \tfrac{1}{2} \cdot \epsilon^C_{\mathcal{D}}(s', \alpha', k', \overline{\alpha}, s_1, s_2, \sigma')
$$

We proceed by induction on $c(s)$. Three subcases.

Case $[s = j]$

$$
\begin{aligned}
&d(\mathcal{D}(j)(\alpha, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma), \mathcal{D}(j)(\alpha, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma))\\
&= d(\sigma \cdot kc([\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \mid \alpha \mapsto \langle j \rangle\,])(\sigma),\\
&\qquad \sigma \cdot kc([\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \mid \alpha \mapsto \langle j \rangle\,])(\sigma))\\
&[\neg(\overline{\alpha} \geq \alpha)]\\
&= \tfrac{1}{2} \cdot d(kc([\, [\, K(k) \mid \alpha \mapsto \langle j \rangle\,] \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma),\\
&\qquad kc([\, [\, K(k) \mid \alpha \mapsto \langle j \rangle\,] \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma)) \qquad \text{[Lemma 5.1]}
\end{aligned}
$$

$= \frac{1}{2} \cdot d(kc([\, K([\, k \mid \alpha \mapsto \langle j \rangle \,]) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \,])(\sigma),$
$\quad kc([\, K([\, k \mid \alpha \mapsto \langle j \rangle \,]) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,])(\sigma))$ $^{(5.10.2.4)}$

And by $^{(5.10.1)}$, $\exists s' \in Stat, \alpha' \in Id, k' \in KRes, \sigma' \in \Sigma$ with $\neg(\overline{\alpha} \geq \alpha')$ such that:

$^{(5.10.2.4)} \leq \frac{1}{2} \cdot d(\mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \,])(\sigma'),$
$\quad \mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,])(\sigma'))$

Case $[s = x]$

$d(\mathcal{D}(x)(\alpha, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \,])(\sigma), \mathcal{D}(x)(\alpha, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,])(\sigma))$
$= d(\mathcal{D}(D(x))(\alpha, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \,])(\sigma),$
$\quad \mathcal{D}(D(x))(\alpha, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,])(\sigma))$ $^{(5.10.2.5)}$

By the induction hypothesis $(c(D(x)) < c(x))$, $\exists s' \in Stat, \alpha' \in Id, k' \in KRes, \sigma' \in \Sigma$ with $\neg(\overline{\alpha} \geq \alpha')$ such that:

$^{(5.10.2.5)} \leq \frac{1}{2} \cdot d(\mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \,])(\sigma'),$
$\quad \mathcal{D}(s')(\alpha', [\, K(k') \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,])(\sigma'))$

Case $[s = s^1 \parallel s^2]$

$\epsilon^C_{\mathcal{D}}(s^1 \parallel s^2, \alpha, k, \overline{\alpha}, s_1, s_2, \sigma)$
$= d(\mathcal{D}(s^1 \parallel s^2)(\alpha, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \,])(\sigma),$
$\quad \mathcal{D}(s^1 \parallel s^2)(\alpha, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,])(\sigma))$
$= d(\mathcal{D}(s^1)(\alpha \cdot 1, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s^2) \,])(\sigma) \cup$
$\quad \mathcal{D}(s^2)(\alpha \cdot 2, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s^1) \,])(\sigma),$
$\quad \mathcal{D}(s^1)(\alpha \cdot 1, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \mid \alpha \cdot 2 \mapsto \mathcal{D}(s^2) \,])(\sigma) \cup$
$\quad \mathcal{D}(s^2)(\alpha \cdot 2, [\, K(k) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \mid \alpha \cdot 1 \mapsto \mathcal{D}(s^1) \,])(\sigma))$

['$\cup$' is nonexpansive; $\neg(\overline{\alpha} \geq \alpha) \Rightarrow \neg(\overline{\alpha} \geq \alpha \cdot 1), \neg(\overline{\alpha} \geq \alpha \cdot 2)$; Lemma 5.1]

$\leq max\{d(\mathcal{D}(s^1)(\alpha \cdot 1, [\, K[\, k \mid \alpha \cdot 2 \mapsto s^2 \,] \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \,])(\sigma),$
$\quad \mathcal{D}(s^1)(\alpha \cdot 1, [\, K[\, k \mid \alpha \cdot 2 \mapsto s^2 \,] \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,])(\sigma))$ $^{(5.10.2.6)}$
$\quad d(\mathcal{D}(s^2)(\alpha \cdot 2, [\, K[\, k \mid \alpha \cdot 1 \mapsto s^1 \,] \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \,])(\sigma),$
$\quad \mathcal{D}(s^2)(\alpha \cdot 2, [\, K[\, k \mid \alpha \cdot 1 \mapsto s^1 \,] \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,])(\sigma))$ $^{(5.10.2.7)}$ $\}$

As $\neg(\overline{\alpha} \geq \alpha \cdot 1)$ and $\neg(\overline{\alpha} \geq \alpha \cdot 2)$, we can apply the induction hypothesis and we infer that $\exists s'_1 \in Stat, \alpha'_1 \in Id, k'_1 \in KRes, \sigma'_1 \in \Sigma$ with $\neg(\overline{\alpha} \geq \alpha'_1)$ and $\exists s'_2 \in Stat, \alpha'_2 \in Id, k'_2 \in KRes, \sigma'_2 \in \Sigma$ with $\neg(\overline{\alpha} \geq \alpha'_2)$, such that:

$^{(5.10.2.6)} \leq \frac{1}{2} \cdot d(\mathcal{D}(s'_1)(\alpha'_1, [\, K(k'_1) \mid \overline{\alpha} \mapsto \mathcal{D}(s_1) \,])(\sigma'_1),$
$\quad \mathcal{D}(s'_1)(\alpha'_1, [\, K(k'_1) \mid \overline{\alpha} \mapsto \mathcal{D}(s_2) \,])(\sigma'_1))$ $^{(5.10.2.6')}$

$$^{(5.10.2.7)} \leq \frac{1}{2} \cdot d(\mathcal{D}(s_2')(\alpha_2', [\, K(k_2') \mid \overline{\alpha} \mapsto \mathcal{D}(s_1)\,])(\sigma_2'),$$
$$\mathcal{D}(s_2')(\alpha_2', [\, K(k_2') \mid \overline{\alpha} \mapsto \mathcal{D}(s_2)\,])(\sigma_2'))\ ^{(5.10.2.7')}$$

Finally, by taking $max\{\ ^{(5.10.2.6')}\ ,\ ^{(5.10.2.7')}\ \}$ we obtain:

$$\epsilon^C_{\mathcal{D}}(s^1 \parallel s^2, \alpha, k, \overline{\alpha}, s_1, s_2, \sigma) \leq max\{\ ^{(5.10.2.6')}\ ,\ ^{(5.10.2.7')}\ \}$$
$$= max\{\tfrac{1}{2} \cdot \epsilon^C_{\mathcal{D}}(s_1', \alpha_1', k_1', \overline{\alpha}, s_1, s_2, \sigma_1'), \tfrac{1}{2} \cdot \epsilon^C_{\mathcal{D}}(s_2', \alpha_2', k_2', \overline{\alpha}, s_1, s_2, \sigma_2')\}$$

This implies immediately the desired result. □

The main results are provided by Theorem 5.11. By using this theorem one can reason in a compositional manner upon the behavior of MCC programs. In general, the properties stated in Theorem 5.11 hold for all continuations containing only computations denotable by program statements. In practice this may be sufficient, because the initial continuation (α_0, κ_0) is empty (contains no computations) and according to the semantic equations, the denotational mapping $\mathcal{D}(\cdot)$ adds only denotations of statements to the continuation. When evaluated with respect to the initial (empty) continuation, the denotational mapping preserves the following invariant property: a CSC continuation contains only computations denotable by program statements. Properties 5.11(a)-(d) hold for all continuations.

THEOREM 5.11 *For all* $s, s_1, s_2, s_3 \in Stat$ *:*

(a) $\mathcal{D}(s_1 + s_2) = \mathcal{D}(s_2 + s_1)$

(b) $\mathcal{D}((s_1 + s_2) + s_3) = \mathcal{D}(s_1 + (s_2 + s_3))$

(c) $\mathcal{D}(s + s) = \mathcal{D}(s)$

(d) $\mathcal{D}((s_1 + s_2); s_3) = \mathcal{D}((s_1; s_3) + (s_2; s_3))$

Also, for all $s_1, s_2, s_3 \in Stat, (\alpha, k), (\overline{\alpha}, \overline{k}) \in CRes, (\alpha, k) \cong (\overline{\alpha}, \overline{k})$ *and for all contexts* C*:*

(e) $\mathcal{D}(C(s_1 \| s_2))(\alpha, K(k)) = \mathcal{D}(C(s_2 \| s_1))(\overline{\alpha}, K(\overline{k}))$

(f) $\mathcal{D}(C(s_1 \| (s_2 \parallel s_3)))(\alpha, K(k)) = \mathcal{D}(C((s_1 \| s_2) \parallel s_3))(\overline{\alpha}, K(\overline{k}))$

(g) $\mathcal{D}(C(s_1; (s_2; s_3)))(\alpha, K(k)) = \mathcal{D}(C((s_1; s_2); s_3))(\overline{\alpha}, K(\overline{k}))$

(h) $\mathcal{D}(C(s_1; (s_2 + s_3)))(\alpha, K(k)) = \mathcal{D}(C((s_1; s_2) + (s_1; s_3)))(\overline{\alpha}, K(\overline{k}))$

Proof: (a), (b) and (c) follow immediately from the properties of the union operator '$\cup$'. For (d) the proof was given at the beginning of section 5 for all continuations and $(\alpha, \kappa) \in \mathbf{Cont}$ and for all states $\sigma \in \Sigma$. Here we only treat (e); (f), (g) and (h) can be handled similarly.

$$\mathcal{D}(C(s_1 \parallel s_2))(\alpha, K(k))(\sigma)$$

$$= \mathcal{D}(C(s_1 \parallel s_2))(\alpha, K(k))(\sigma) \qquad [\text{ Lemma 5.8(a), Lemma 5.9}]$$

$$= \mathcal{D}(C(s_2 \parallel s_1))(\alpha, K(k))(\sigma) \qquad [\text{ Corollary 1(a)}]$$

$$= \mathcal{D}(C(s_2 \parallel s_1))(\overline{\alpha}, K(\overline{k}))(\sigma)$$

which implies $\mathcal{D}(C(s_1 \parallel s_2))(\alpha, K(k)) = \mathcal{D}(C(s_2 \parallel s_1))(\overline{\alpha}, K(\overline{k}))$, because the proof is given for an arbitrary σ. □

The function $\mathcal{D}[\![\cdot]\!]$ (see Definition 4.1) is *not* defined in a compositional manner; only $\mathcal{D}(\cdot)$ is a compositional (denotational) mapping. However, $\mathcal{D}[\![s]\!]$ is useful because it evaluates a (whole) program s with respect to the initial (empty) continuation. The following corollary can be used to reason about the behavior of $\mathcal{D}[\![s]\!]$.

Corollary 3 *For all $s, s_1, s_2, s_3 \in Stat$ and for all contexts C:*

(a) $\mathcal{D}[\![C(s_1 + s_2)]\!] = \mathcal{D}[\![C(s_2 + s_1)]\!]$

(b) $\mathcal{D}[\![C((s_1 + s_2) + s_3)]\!] = \mathcal{D}[\![C(s_1 + (s_2 + s_3))]\!]$

(c) $\mathcal{D}[\![C(s + s)]\!] = \mathcal{D}[\![C(s)]\!]$

(d) $\mathcal{D}[\![C((s_1 + s_2); s_3)]\!] = \mathcal{D}[\![C((s_1; s_3) + (s_2; s_3))]\!]$

(e) $\mathcal{D}[\![C(s_1 \parallel s_2)]\!] = \mathcal{D}[\![C(s_2 \parallel s_1)]\!]$

(f) $\mathcal{D}[\![C(s_1 \parallel (s_2 \parallel s_3))]\!] = \mathcal{D}[\![C((s_1 \parallel s_2) \parallel s_3)]\!]$

(g) $\mathcal{D}[\![C(s_1; (s_2; s_3))]\!] = \mathcal{D}[\![C((s_1; s_2); s_3)]\!]$

(h) $\mathcal{D}[\![C(s_1; (s_2 + s_3))]\!] = \mathcal{D}[\![C((s_1; s_2) + (s_1; s_3))]\!]$

Proof: It suffices to consider one case.

$$\mathcal{D}[\![C(s_1; (s_2; s_3))]\!](\sigma)$$

$$= \mathcal{D}(C(s_1; (s_2; s_3)))(\alpha_0, \kappa_0)(\sigma) \qquad [\text{Theorem 5.11(g)}]$$

$$= \mathcal{D}(C((s_1; s_2); s_3))(\alpha_0, \kappa_0)(\sigma)$$

$$= \mathcal{D}[\![C((s_1; s_2); s_3)]\!](\sigma)$$

which implies $\mathcal{D}[\![C(s_1; (s_2; s_3))]\!] = \mathcal{D}[\![C((s_1; s_2); s_3)]\!]$, because the proof is given for an arbitrary σ. □

6 Concluding Remarks

The first steps toward a continuation semantics for concurrency (CSC) are presented in [13]. Our previous work suggests that the (CSC) technique can provide a very good flexibility in the denotational design of parallel and distributed systems. We developed denotational models designed with CSC for a couple of advanced control concepts like the nondeterministic promotion in Andorra-like languages [15], and the communication with synchronization on multiple channels [14] in the style of Join calculus [7]. As far as we know these concepts have not been modeled denotationally until now without CSC. But the distinctive characteristic of the CSC technique is the representation of continuations as structures of computations rather than the straightforward functions to some answer type that are used in the classic technique of continuations.

This paper describes the concurrent systems in denotational models designed with CSC by using standard techniques from metric semantics [3]. We illustrate these ideas on the particular example of a CSP-like language MCC extended with a mechanism for synchronization on multiple channels in the style of Join calculus [7]. For such a language we have proved that the semantic operators designed with continuations obey the basic laws given in the algebraic theories of concurrency, such as the associativity and the commutativity of parallel composition.

The basic idea of the proofs was to show that the properties under consideration are preserved by the computation steps. We also used an '$\epsilon \leq \frac{1}{2} \cdot \epsilon \Rightarrow \epsilon = 0$' argument which is standard in metric semantics [3]. The identification of the semantic properties from the invariants of the computation is common in classic bisimulation semantics [10].

As further work, we think to use the elegance of the semantic domains in expressing the CSC technique. In this paper the domain of continuations was modeled with the aid of a function space from a set of identifiers (endowed with a partial order) to the domain of computations: $Id \rightarrow \mathbf{Comp}$. According to Corollary 1, any two isomorphic continuations behave the same. Intuitively, the domain of continuations could be defined in terms of isomorphism classes $[Id \rightarrow \mathbf{Comp}]$ of such structures. Since the existing metric models based on isomorphism classes of semantic structures (in particular the metric pomset model [4]) do not involve domains defined by reflexive equations (like $\mathbf{Comp}$), such a construction requires further work.

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